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1962

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Abstract

Full Text

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ON THE PROBLEM OF BRANCHING OF SOLUTIONS OF A NONLINEAR ANALYTIC EQUATION

(Presented by Academician S. L. Sobolev on 10 III 1962)

In the complex Banach space E consider the equation

$$\varphi = A\varphi, \tag{1}$$

where the operator A is analytic and completely continuous in some open domain G of the space E. From the results of Cronin (see (^{1}), Theorem 5.1, pp. 228–230, and (^{2}), Theorem B, pp. 177–180) the following assertion follows immediately:

Theorem 1. Let φ_0 ($\varphi_0 \in G$) be an isolated solution of equation (1), and let γ be the index of this solution. Then

$$\gamma > 0, \tag{2}$$

and in the case when unity is an eigenvalue of the Fréchet derivative of the operator A at the point φ_0,

$$\gamma > 1. \tag{3}$$

In the present article Theorem 1 is applied to the study of the question of branching of solutions of the equation

$$\varphi = A_\mu \varphi, \tag{4}$$

where the operator $A_\mu \varphi$, analytic and completely continuous with respect to φ, depends analytically on the complex parameter μ.

1. Suppose that $A0 = 0$ ($0 \in G$), and consider the question of bifurcation points of the operator A.

Recall (see (^{3})) that the number μ_0 is called a bifurcation point of the operator A if, for every $\varepsilon > 0$, there is a characteristic number μ of the operator A such that $|\mu - \mu_0| < \varepsilon$, and to this characteristic number there corresponds at least one eigenfunction φ: $\varphi = \mu A\varphi$, with norm less than ε.

Denote by B the Fréchet derivative of the operator A at the point 0. It is not difficult to see that the bifurcation points of the operator A can only be the characteristic numbers of the operator B (see (^{3}), pp. 195–196).

M. A. Krasnosel' skii showed in (^{3}) that a characteristic number μ_0 of the operator B is a bifurcation point of the operator A (not necessarily analytic) if the zero solution of the equation

$$\varphi = \mu_0 A\varphi \tag{5}$$

is isolated and the index of this solution is not equal in absolute value to unity.

From this result of M. A. Krasnosel' skii and Theorem 1 it follows:

Theorem 2. The bifurcation points of the operator A coincide with the characteristic numbers of the operator B.

We note that for nonanalytic operators Theorem 2 is not true.

2. In this and the following sections it is assumed that the space E is a complex Banach space with a basis (for example, a Hilbert space).

Let D be some bounded connected domain of the space E; L the boundary of D, and $D + L \in G$. The theorem below for the finite-dimensional case was proved in (^{2}).

Theorem 3. *Suppose the vector field $\varphi - A\varphi$ does not vanish on L.*

Then the rotation γ of this field on L is nonnegative.

From Theorems 1 and 3 it follows that inside L there can be no more than γ isolated solutions of equation (1). It turns out that equation (1) has no non-isolated solutions in the domain D.

Theorem 4. *Suppose the conditions of Theorem 3 are satisfied.*

Then equation (1) has a finite number of solutions in the domain D.

The proof is based on reduction to the n-dimensional case.

For $n = 2$, Theorem 4, in a somewhat different formulation, is presented in (^{4}) (see p. 151).

3. Let φ_0 ($\varphi_0 \in G$) be an isolated solution of equation (1). Denote by S such a sphere with center at the point φ_0 that in the ball T bounded by it equation (1) has no solutions other than φ_0.

Let

$$\alpha = \min_{\varphi \in S} \|\varphi - A\varphi\|. \tag{6}$$

From Theorems 1 and 4 it obviously follows that

Theorem 5. *If a completely continuous operator A_1, analytic in the ball T, satisfies, for $\varphi \in S$, the inequality*

$$\|A_1\varphi - A\varphi\| < \alpha, \tag{7}$$

then the equation

$$\varphi = A_1\varphi \tag{8}$$

has in the ball T a finite (but nonzero) number of solutions.

4. We now consider the equation depending on the complex parameter μ,

$$\varphi = B\varphi + \mu f + \sum_{l+k\geq 2}^{\infty} \mu^l A_{lk}\varphi^k, \tag{9}$$

where f is some element of E; B is a linear completely continuous operator, $A_{lk}\varphi^k$ is a homogeneous form of degree k, and the series $\sum_{l+k\geq 2}^{\infty} \mu^l A_{lk}\varphi^k$ converges for $|\mu| \leq \varepsilon_1$, $\|\varphi\| \leq \varepsilon_2$, uniformly with respect to μ, φ.

In seeking small solutions of equation (9), two cases are possible:

1) Unity is not an eigenvalue of the operator B. Then, for sufficiently small values of μ, equation (9) has a unique small solution, and it is representable in the form

$$\varphi = \sum_{i=1}^{\infty} \mu^i \varphi_i \quad (\varphi_1, \varphi_2, \ldots \in E). \tag{10}$$

2) Unity is an eigenvalue of the operator B of multiplicity n. Then to equation (9) we apply the Lyapunov-Schmidt method (see, for example, (^{5})) of reduction to the branching equations

$$\psi_i(c_1, \ldots, c_n, \mu) = 0 \quad (i = 1, \ldots, n;\ c_1, \ldots, c_n \text{ are numerical parameters}), \tag{11}$$

and every small solution of equation (9) can be obtained from some series

$$\varphi = \sum_{r_0+r_1+\cdots+r_n\geq 1}^{\infty} h_{r_0,r_1,\ldots,r_n} \mu^{r_0} c_1^{r_1} \cdots c_n^{r_n} \quad (h_{r_0,r_1,\ldots,r_n} \in E) \tag{12}$$

by substituting into it a small solution $c_1, \ldots, c_n, \mu$ of system (11).

Denote by H the manifold of solutions of system (11). The manifold H is a certain analytic manifold in the $(n+1)$-dimensional complex space R_{n+1} with coordinates $c_1, \dots, c_n, \mu$. In some neighborhood of the origin of the coordinates of the space R_{n+1}, the manifold H can be represented uniquely in the form of the sum of a finite number of irreducible manifolds analytic at the origin of coordinates (see (^{6})):

$$H = H_1 + \cdots + H_m. \tag{13}$$

Suppose further that for $\mu = 0$ the zero solution of equation (9) is isolated. In this case, by means of Theorem 5 it is established that the dimension of each manifold H_j of the decomposition (13) is equal to one. Thus, the coordinates $c_1, \dots, c_n, \mu$ of the points of H_j $(j = 1, \dots, m)$ are analytic functions of some parameter t:

$$c_1 = c_{1j}(t), \dots, c_n = c_{nj}(t), \quad \mu = \mu_j(t) \quad (c_{ij}(0) = \mu_j(0) = 0,\ \mu_j(t) \neq 0). \tag{14}$$

From formulas (12), (14) it follows that the small solutions of equation (9) are representable by the series

$$\varphi = \varphi_j(\mu) = \sum_{j=1}^{\infty} \varphi_{ij}(\mu)^{i/s_j} \quad (j = 1, \dots, m;\ \varphi_{ij} \in E;\ s_j \text{ are natural numbers}). \tag{15}$$

Let us formulate the result obtained:

Theorem 6. *Suppose that for $\mu = 0$ the zero solution of equation (9) is isolated.*

Then there exist positive numbers δ_1, δ_2 such that, for $|\mu| < \delta_1$, equation (9) has in the ball of radius δ_2 with center at zero a finite number of solutions.

Each of these solutions is representable in the form (15).

We note that in the case where unity is a simple eigenvalue of the operator B, the representability of small solutions of equation (9) by series (15) was proved by V. V. Pokornyi (^{7}).

For the case where unity is an eigenvalue of the operator B of multiplicity 2, the assertion of Theorem 6 was obtained by another method jointly by P. P. Rybin and the author (unpublished).

The author expresses gratitude to M. A. Krasnosel' skii for a number of suggestions.

Received
27 II 1962

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Note: Figure translations are in progress. See original paper for figures.

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