I. F. KUSHNIRCHUK

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Abstract

Full Text

I. F. KUSHNIRCHUK

ON A CERTAIN PARTIAL DIFFERENTIAL EQUATION OF HIGHER ORDER

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Introduction. In the paper ([1]), M. K. Fage solves the Cauchy problem with respect to the real variable x for the equation

$$-\frac{\partial^n f}{\partial w^n} + \frac{\partial^n f}{\partial x^n} + \sum_{k+m\le n-1} p_{km}(w,x)\frac{\partial^{k+m} f}{\partial w^k \partial x^m} = h(w,x) \tag{1}$$

with initial conditions

$$f(w,0) = f_0(w), \qquad \left.\frac{\partial f}{\partial x}\right|_{x=0} = f_1(w), \dots, \left.\frac{\partial^{n-1} f}{\partial x^{n-1}}\right|_{x=0} = f_{n-1}(w), \tag{2}$$

where the functions $p_{km}(w,x)$ $(0 \le k+m \le n-1)$ and $h(w,x)$ are continuous jointly in the variables w and x in the cylinder $C = G \times [0,b)$ of the complex-real space $\tilde{E}_3$ and, together with $f_k(w)$ $(k = 0,1,2,\dots,n-1)$, are analytic in $w = u + iv$ in the simply connected domain G.

This solution is obtained by reducing equation (1) to Bianchi' s equation (see ([2]))

$$(-1)^n \frac{\partial^n \mathcal{F}}{\partial t_1 \dots \partial t_n} + \sum_{|K|\le n-1} q_K(t) D_K \mathcal{F} = \mathcal{H}(t), \tag{3}$$

which is achieved by passing from the variables w and x to the characteristic variables $t = (t_1, t_2, \dots, t_n)$ according to the formulas

$$w = w_0 + \sum_{k=1}^{n} \varepsilon_k t_k, \qquad x = x_0 - \sum_{k=1}^{n} t_k, \tag{4}$$

where w_0, x_0 are the coordinates of some point $P_0 \in C$, and ε_k $(k = 1,2,\dots,n)$ are the n-th roots of unity.

If $t_1 \ge 0, t_2 \ge 0, \dots, t_n \ge 0;$ $t_1 + t_2 + \dots + t_n \le x_0$, then the point $P(w,x)$, whose coordinates are determined by formulas (4), describes the characteristic pyramid (hereafter abbreviated c.p.) V_{P_0} of equation (1) with vertex $P_0(w_0, x_0)$ and base

W_{w_0,x_0}, which is the regular n-gon with vertices at the points $P_k(w_0+\varepsilon_k x_0, 0)$ $(k=1,2,\dots,n)$. Consequently, in this case a unique c.p. corresponds to each point. A point $P' \in C$ is called reachable with respect to the domain G if the base W of the pyramid $V_{P'}$ belongs to G.

In the present note, the Cauchy problem is solved for the equation

$$\sum_{k=0}^{n} p_k(x)\frac{\partial^n f}{\partial w^k \partial x^{n-k}} + \sum_{l+m\leq n-1} p_{lm}(w,x)\frac{\partial^{l+m} f}{\partial w^l \partial x^m} = h(w,x) \tag{5}$$

with initial conditions (2), where $p_k(x)$ $(k=1,2,\dots,n;\ p_0(x)\equiv 1)$ are functions continuous on $[0,b)$ $(b>0)$.

It turns out that in the case of equation (5), to each reachable point of the cylinder C there correspond, generally speaking, 2^n c.p.' s (see the example in § 2).

Therefore, in § 3 the uniqueness conditions of the c.p. are first established, and in § 4 the Cauchy problem (5), (2) is solved under these conditions.

To simplify the notation, all subsequent calculations are carried out for an equation of the 3rd order $(n=3)$.

§ 1. Characteristic pyramids. We shall assume that the roots $\lambda_k(x) = -a_k(x) - ib_k(x)$ $(k=1,2,3)$ of the characteristic equation

$$\lambda^3 + p_1(x)\lambda^2 + p_2(x)\lambda + p_3(x) = 0 \tag{6}$$

for equation (5) are pairwise distinct. The characteristics of equation (5) are determined from the ordinary differential equations of the 1st order

$$\frac{dw}{dx} = -\lambda_k(x) \qquad (k=1,2,3)$$

or the equivalent system of two equations

$$\frac{du}{dx} = a_k(x), \qquad \frac{dv}{dx} = b_k(x). \qquad (k=1,2,3). \tag{7}$$

Let $P_0(w_0, x_0)$ be an interior point of the cylinder . Draw through it the three characteristics of equation (5):

$$u = u_0 + \int_{x_0}^{x} a_k(\xi)\,d\xi, \qquad v = v_0 + \int_{x_0}^{x} b_k(\xi)\,d\xi \qquad (k=1,2,3)$$

and continue them until they meet the plane $w(x=0)$ at the points $P_k(w_k, 0)$, where $w_k = u_k + iv_k$,

$$u_k = \int_{x_0}^{0} a_k(\xi)\, d\xi + u_0, \qquad v_k = \int_{x_0}^{0} b_k(\xi)\, d\xi + v_0.$$

Next construct the surface S_{kl} $(k \neq l)$ by means of the characteristics of the k-th family issuing from the points of the curve P_0P_k. The surfaces S_{kl} and S_{lk}, having the common lines P_0P_k and P_0P_l, generally speaking, will not coincide (see § 2). Any three of these surfaces together with the plane w bound a certain part of the space $\widetilde{E}_3$, which it is natural to call a c.p. of equation (5). The number of such pyramids is

$$\sum_{m=0}^{3} \binom{3}{m} = 2^3.$$

§2. Example. Let the group of highest terms of an equation of the 3rd order have the form

$$\frac{\partial^3 f}{\partial x^3} + (2x+3)(1+i)\frac{\partial^3 f}{\partial w\, \partial x^2} + (4x^2+6x+7)i\frac{\partial^3 f}{\partial w^2 \partial x} +$$

$$+2[-4x^2 + x - 2 + (2x^2+2x+1)i]\frac{\partial^3 f}{\partial w^3}.$$

Then the roots of the corresponding equation (6) are the functions $\lambda_1 = -(1+2i)$, $\lambda_2 = -(2x+i)$, $\lambda_3 = -(2+2xi)$. Taking the point P_0 with $w_0 = 0$ and $x = 1$, we write the equations of the surfaces:

$$S_{12}: \; u - v^2 - 3v + 4xv - 4x^2 + 5x - 1 = 0,$$

$$S_{21}: \; u + v^2 + 3v - 2xv - 3x + 3 = 0,$$

$$S_{13}: \; u^2 + 2u - v - 2xu + x^2 - 1 = 0,$$

$$S_{31}: \; u^2 + 4u + v - 4xu + 3x^2 - 8x + 5 = 0,$$

$$S_{23}: \; u + v \pm \sqrt{-u + x^2 - 1} - x^2 - x + 2 = 0,$$

$$S_{32}: \; u + v \pm \sqrt{2 + u - 2x - x^2} - 2x + 3 = 0.$$

Fig. 1

Figure 1: Fig. 1

In Fig. 1 the lines of intersection of these surfaces with the plane w are shown; combining them, we obtain 8 contours of the bases of the corresponding c. s.

§ 3. **Uniqueness conditions for c. s.** To find the uniqueness conditions for a c. s. with vertex $P_0(w_0, x_0) \equiv P_0(u_0, v_0, x_0)$ (which, evidently, reduce to the conditions of coincidence of the surfaces S_{kl} and S_{lk} for all k and l, $k \neq l$), we consider first-order partial differential equations

$$A_k(\varphi) \equiv \frac{\partial \varphi}{\partial x} + a_k(x)\frac{\partial \varphi}{\partial u} + b_k(x)\frac{\partial \varphi}{\partial v} = 0 \quad (k = 1, 2, 3), \tag{8}$$

for each of which the system (7) (with the same k) is characteristic.

Theorem 1. If the system of equations

$$A_k(\varphi) = 0, \qquad A_l(\varphi) = 0, \qquad k \neq l, \tag{9}$$

has a solution $\varphi = \varphi(u, v, x)$, different from an identically constant one and satisfying the condition $\varphi(u_0, v_0, x_0) = 0$, then the surfaces S_{kl} and S_{lk} coincide.

In the following theorem we clarify the conditions under which system (9) has the above-mentioned solution (see ([3]), p. 367).

Theorem 2. In order that system (9) have a solution $\varphi = \varphi(u, v, x)$, different from an identically constant one, it is necessary and sufficient that the determinant

$$\mathcal{D}_{kl} \equiv \begin{vmatrix} 1 & a_k & b_k \\ 1 & a_l & b_l \\ 0 & A_{kl} & B_{kl} \end{vmatrix},$$

be equal to zero (identically with respect to x), where

$$A_{kl} = A_k(a_l) - A_l(a_k), \qquad B_{kl} = A_k(b_l) - A_l(b_k).$$

Fig. 1

From Theorem 2 it follows immediately:

Theorem 3. For the existence of a unique c. s. V_{P_0} with vertex at an attainable point $P_0 \in \Omega$, it is necessary and sufficient that, for all $k \neq l$,

$$\mathcal{D}_{kl} \equiv 0. \tag{10}$$

Remark 1. The sufficiency of the more restrictive conditions

$$A_{kl} \equiv 0, \qquad B_{kl} \equiv 0 \tag{11}$$

for the coincidence of the surfaces S_{kl} and S_{lk} was obtained by another method in (4).

Remark 2. In the example given above, $\mathcal{D}_{12} = 2,\ \mathcal{D}_{13} = 2,\ \mathcal{D}_{23} = 6$.

§ 4. **Solution of the Cauchy problem (5), (2).** It can be shown that, by a change of the independent variable x, equation (5), for which conditions (10) are satisfied, is reduced to an equation of the same form, for which conditions (11) hold. We shall therefore suppose that conditions (11) hold. Then the real and imaginary parts of the roots of equation (6), taken with the opposite sign, have the form

$$a_k(x) = a(x) + \alpha_k, \qquad b_k(x) = b(x) + \beta_k \quad (k = 1, 2, 3).$$

with constant $\alpha_k,\ \beta_k$, whence

$$-\lambda_k(x) = a(x) + ib(x) + \alpha_k + i\beta_k = c(x) + \gamma_k \qquad (k = 1, 2, 3).$$

With an attainable point $P_0(w_0, x_0)$ of the cylinder C we associate the characteristic variables t_1, t_2, t_3 by the formulas

$$w = w_0 + \int_{x_0}^{x} c(\xi)\, d\xi - \gamma_1 t_1 - \gamma_2 t_2 - \gamma_3 t_3, \qquad x = x_0 - t_1 - t_2 - t_3. \tag{12}$$

Consider, in the space E_3 of the variables t_1, t_2, t_3, the pyramid

$$\Gamma = (t_1 \geq 0,\ t_2 \geq 0,\ t_3 \geq 0;\ t_1 + t_2 + t_3 \leq x_0)$$

with base

$$\Omega = (t_1 \geq 0,\ t_2 \geq 0,\ t_3 \geq 0;\ t_1 + t_2 + t_3 = x_0).$$

Then the following theorem holds, analogous to Theorem 2 from (1).

Theorem 4. *By means of formulas (12), the pyramid Γ is transformed into the characteristic parallelepiped V_{P_0} of equation (5), and its base Ω into the base W of the pyramid V_{P_0}, which is the triangle with vertices $P_k(u_k, v_k, 0)$, where*

$$u_k = u_0 + \int_{x_0}^{0} a(\xi)\, d\xi - \alpha_k x_0, \qquad v_k = v_0 + \int_{x_0}^{0} b(\xi)\, d\xi - \beta_k x_0, \qquad k = 1, 2, 3.$$

The following theorem resolves the question of reducing the given equation (5) to the Bianchi equation.

Theorem 5. *If the roots* $\lambda_k(x)$ *possess a continuous derivative of second order on* $[0, b)$ *(equal to* $-c''(x)$*), then equation (5), by means of formulas (12), is reduced to the Bianchi equation (3) for* $n = 3$ *(with continuous coefficients in* Γ*) with respect to* $\mathcal{F}(t_1, t_2, t_3)$ *and right-hand side* $\mathcal{H}(t_1, t_2, t_3)$*, obtained from* $f(w, x)$*,* $h(w, x)$ *by the substitution (12), and the initial conditions (2) to the following limiting conditions on* Ω*:*

$$\mathcal{F}(t_1, t_2, t_3)\big|_\Omega = f_0(w); \qquad \frac{\partial \mathcal{F}}{\partial t_k}\bigg|_\Omega = -f_1(w) + \lambda_k(0) f_0'(w);$$

$$\frac{\partial^2 \mathcal{F}}{\partial t_k \partial t_l}\bigg|_\Omega = f_2(w) - (\lambda_k(0) + \lambda_l(0)) f_1'(w) + \lambda_k(0)\lambda_l(0) f_0''(w) - \lambda_k'(0) f_0'(w),$$

in the right-hand sides of which

$$w = w_0 + \int_{x_0}^{0} c(\xi)\, d\xi - \gamma_1 t_1 - \gamma_2 t_2 - \gamma_3 t_3, \qquad t_1 + t_2 + t_3 = x_0.$$

By the method of successive approximations one proves:

Theorem 6. *Let* P_0 *be an attainable point of the cylinder* C*. Then there exists a unique function* $f(w, x)$ *satisfying equation (5) in the characteristic parallelepiped* V_{P_0} *and the initial conditions (2) on its base* W*.*

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Chernivtsi State University

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