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Abstract

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MATHEMATICS

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ON HOMOGENEOUS GRID SCHEMES

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A. N. Tikhonov and A. A. Samarskii (see, for example, the survey paper ([1])) singled out, among difference schemes corresponding to differential equations with one spatial variable, the so-called homogeneous difference schemes, which make it possible to carry out "through computation" in the case of equations with discontinuous coefficients. In the present note we shall consider, by a somewhat different route, homogeneous grid schemes corresponding to differential equations with two spatial variables.

1°. For simplicity let us take the Poisson equation

$$\lambda \Delta U \equiv \lambda \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = f, \qquad \lambda > 0, \tag{1}$$

where the coefficients λ and f are piecewise constant. On the lines Γ on which the constancy of the coefficient λ is violated, the usual conjugation conditions must be satisfied

$$[U]_- = [U]_+, \qquad \left[\lambda \frac{\partial U}{\partial \nu}\right]_- = \left[\lambda \frac{\partial U}{\partial \nu}\right]_+, \tag{2}$$

where the minus and plus signs mean that the quantities enclosed in square brackets are taken on different sides of Γ; ν is the direction of the normal to Γ.

Fig. 1

Fig. 1

In what follows we shall assume that the lines of discontinuity of the coefficients always pass through grid nodes. If, at the grid nodes at which λ is continuous, one approximates the given equation (1), and at the remaining nodes (where equation (1) has no meaning: the first and second derivatives of U do not exist) one approximates the conjugation conditions (2), then a nonhomogeneous grid scheme is obtained. More convenient for programming, and—most importantly—considerably more accurate are homogeneous grid schemes, in which the same grid equations are used irrespective of whether or not the coefficients λ and f are discontinuous at the given node.

2°. Let us first consider a rectangular grid. Introduce the following notation (see Fig. 1): A_i $(i = 0, 1, \dots, 8)$ are respectively the points (x, y), $(x + h_1, y)$, $(x, y+h_2)$, $(x-h_3, y)$, $(x, y-h_4)$, $(x+h_1, y+h_2)$, $(x-h_3, y+h_2)$, $(x-h_3, y-h_4)$, $(x+h_1, y-h_4)$; u_i are the values of the function u at the points A_i $(i = 0, 1, \dots, 8)$; Π_i $(i = 1, 2, 3, 4)$ are respectively the pentagons $A_0B_iP_iQ_iB_{i+1}$ $(B_5 = B_1)$; π_i are the intersections of the boundaries of the pentagons Π_i $(i = 1, 2, 3, 4)$ with the boundary of the polygon $B_1P_1Q_1B_2P_2Q_2B_3P_3Q_3B_4P_4Q_4$; λ_i are the values of the coefficient λ, respectively, in the rectangles $A_0A_iA_{i+4}A_{i+1}$ $(i = 1, 2, 3, 4;$ for $i = 4,$ $A_5 = A_1)$; f_i $(i = 1, 2, 3, 4)$ are analogous notations for f; $\alpha = A_0R_i/A_0A_{i+4}$, $1 \geq \gamma = A_0B_i/h_i$

$(i = 1, 2, 3, 4)$; $\lambda_0 = \lambda_4,$ $h_0 = h_4,$ $h_5 = h_1$. We shall assume that the segments P_iQ_i are perpendicular to the straight lines A_0A_{i+4} $(i = 1, 2, 3, 4)$.

In order not to distinguish the case when all or some of the segments A_0A_i $(i = 1, 2, 3, 4)$ are discontinuity lines of the coefficient λ, from the case when the coefficient λ is smooth on all these lines, we pass from the differential equation (1) with the conjugation conditions (2) to the integral relation

$$\sum_{i=1}^{4} \int_{\pi_i} \lambda_i \frac{\partial U}{\partial \nu}\, ds = \sum_{i=1}^{4} \iint_{\Pi_i} f_i\, dx\, dy, \tag{3}$$

where ν is the direction of the outward normal to π_i.

Relation (3) is obtained by integrating equation (1) over Π_i, applying Green's formula, and summing over i $(i = 1, 2, 3, 4)$ the relations obtained, taking account of the conditions (2).

The required difference equation

$$\begin{aligned} &\frac{\gamma(h_1+h_3)(h_2+h_4)}{4\alpha} L_h u_0 \equiv \\ &\equiv \frac{1}{2}\sum_{i=1}^{4}\left\{\lambda_{i-1}\left[\frac{h_{i-1}}{h_i} - \left(\frac{\gamma}{\alpha}-1\right)\frac{h_i}{h_{i-1}}\right] + \lambda_i\left[\frac{h_{i+1}}{h_i} - \left(\frac{\gamma}{\alpha}-1\right)\frac{h_i}{h_{i+1}}\right]\right\} u_i + \\ &+ \frac{1}{2}\left(\frac{\gamma}{\alpha}-1\right)\sum_{i=1}^{4}\lambda_i\left(\frac{h_i}{h_{i+1}} + \frac{h_{i+1}}{h_i}\right) u_{i+4} - \frac{1}{2}\sum_{i=1}^{4}\lambda_i\left(\frac{h_i}{h_{i+1}} - \frac{h_{i+1}}{h_i}\right) u_0 = \\ &= \frac{1}{2\alpha}\sum_{i=1}^{4} h_i h_{i+1}\left[\gamma^2 - \frac{1}{2}\left(\frac{h_i}{h_{i+1}} + \frac{h_{i+1}}{h_i}\right)^2 (\gamma-\alpha)^2\right] f_i \end{aligned} \tag{4}$$

is obtained by applying to (3) the simplest approximation formulas

$$\int_{\pi_i} \lambda_i \frac{\partial U}{\partial \nu}\, ds = \lambda_i \left[\int_{B_i}^{P_i} \frac{\partial U}{\partial \nu}\, ds + \int_{P_i}^{Q_i} \frac{\partial U}{\partial \nu}\, ds + \int_{Q_i}^{B_{i+1}} \frac{\partial U}{\partial \nu}\, ds \right] \approx$$
$$\approx \lambda_i \left(\frac{U_i - U_0}{h_i}\, \overline{B_i P_i} + \frac{U_{i+4} - U_0}{\sqrt{h_i^2 + h_{i+1}^2}}\, \overline{P_i Q_i} + \frac{U_{i+1} - U_0}{h_{i+1}}\, \overline{Q_i B_{i+1}} \right),$$

$$\iint_{\Pi_i} f_i\, dx\, dy = f_i \left[\gamma^2 h_i h_{i+1} - \frac{(h_i^2 + h_{i+1}^2)^2 (\gamma - \alpha)^2}{2 h_i h_{i+1}} \right],$$

where

$$\overline{B_i P_i} = \gamma h_{i+1} - \frac{h_i^2 + h_{i+1}^2}{h_{i+1}} (\gamma - \alpha), \qquad \overline{Q_i B_{i+1}} = \gamma h_i - \frac{h_i^2 + h_{i+1}^2}{h_i} (\gamma - \alpha),$$

$$\overline{P_i Q_i} = \frac{(h_i^2 + h_{i+1}^2)^{3/2}}{h_i h_{i+1}} (\gamma - \alpha), \qquad i = 1, 2, 3, 4$$

(in the relations written above, for $i = 4$ one should understand $B_5 = B_1,\ U_5 = U_1$).

In order that the coefficients of u_i $(i = 1, 2, \dots, 8)$ in equation (4) be nonnegative, it is sufficient that α satisfy the relations

$$\gamma \geq \alpha \geq \frac{\gamma \max\{h_i^2, h_{i+1}^2\}}{h_i^2 + h_{i+1}^2} \qquad (i = 1, 2, 3, 4). \tag{5}$$

For $\lambda_i \equiv 1,\ f_i \equiv f$ (the case of smooth coefficients), $\alpha = 15/34,\ \gamma = 9/17,\ h_1 = h_3 = h,\ h_2 = h_4 = l$, equation (4) becomes the nine-point equation on a regular rectangular grid

$$\left(\frac{l}{h} - \frac{h}{5l} \right) (u_1 + u_3) + \left(\frac{h}{l} - \frac{l}{5h} \right) (u_2 + u_4) + \frac{1}{10} \left(\frac{h}{l} + \frac{l}{h} \right) (u_5 + u_6 + u_7 + u_8) -$$
$$- 2 \left(\frac{h}{l} + \frac{l}{h} \right) u_0 = \frac{12}{85} (h^2 + l^2) \left(\frac{9hl}{h^2 + l^2} - \frac{h^2 + l^2}{8hl} \right) f. \tag{6}$$

Equation (6) with $f \equiv 0$ (the Laplace equation) is described in [2], where it is shown that it has the highest accuracy in the class of nine-point difference equations. Therefore, in the case of nine ($\alpha \neq \gamma$) nodes it is expedient to set in (4) $\alpha = 15/34$, $\gamma = 9/17$, since the nine-point difference equation obtained in this way will be best in the sense that, for $\lambda_i \equiv 1$, $f_i \equiv f$, $h_i = h_{i+2}$, it passes into the best (see [2]) difference equation (6). At the same time, for the conditions (5) to be fulfilled it is sufficient that the inequalities $h_{i+1} \leq h_i \leq \sqrt{5}\, h_{i+1}$ or $h_i \leq h_{i+1} \leq \sqrt{5}\, h_i$ ($i = 1, 2, 3, 4$) hold.

Let us take, as a measure of the local error R_0 at the node (x, y) of the difference operator L_h on some class of functions U, the quantity

$$|R_0| \equiv \left| L_h U_0 - \frac{1}{(h_1 + h_3)(h_2 + h_4)} \sum_{i=1}^{4} h_i h_{i+1} (\lambda \Delta U)_i \right|;$$

where $(\lambda \Delta U)_i$ is the limiting value of the differential operator $\lambda \Delta U$ at the node (x, y) from the side of the pentagon Π_i ($i = 1, 2, 3, 4$). Then, if the function U has bounded, in $A_0, A_i, A_{i+4} A_{i+1}$ ($i = 1, 2, 3, 4$; for $i = 4$, $A_5 = A_1$), one-sided derivatives of third order, satisfies (2), and (for $\alpha \neq \gamma$)

$$\left(\lambda \frac{\partial^2 U}{\partial x\, \partial y}\right)_1 = \left(\lambda \frac{\partial^2 U}{\partial x\, \partial y}\right)_2 = \left(\lambda \frac{\partial^2 U}{\partial x\, \partial y}\right)_3 = \left(\lambda \frac{\partial^2 U}{\partial x\, \partial y}\right)_4, \tag{7}$$

then $|R_0| \leq O(h)$. If, however, U has bounded one-sided derivatives of fourth order, satisfies (1), (2), (7),

$$\left(\lambda \frac{\partial^3 U}{\partial x^2 \partial y}\right)_4 = \left(\lambda \frac{\partial^3 U}{\partial x^2 \partial y}\right)_1, \quad \left(\lambda \frac{\partial^3 U}{\partial x \partial y^2}\right)_1 = \left(\lambda \frac{\partial^3 U}{\partial x \partial y^2}\right)_2,$$
$$\left(\lambda \frac{\partial^3 U}{\partial x^2 \partial y}\right)_2 = \left(\lambda \frac{\partial^3 U}{\partial x^2 \partial y}\right)_3, \quad \left(\lambda \frac{\partial^3 U}{\partial x \partial y^2}\right)_3 = \left(\lambda \frac{\partial^3 U}{\partial x \partial y^2}\right)_4,$$

and $h_1 = h_3$, $h_2 = h_4$, or $\alpha = \frac{2}{3}\gamma$, then $|R_0| \leq O(h^2)$. Thus, as another reasonable choice of the parameters α and γ for $h_1 \neq h_3$ or $h_2 \neq h_4$, one may recommend the values $\alpha = \frac{3}{7}$, $\gamma = \frac{9}{14}$ $\left(\alpha = \frac{2}{3}\gamma\right)$. These values of the parameters α and γ are reasonable in the sense that, for $\lambda_i \equiv 1$, $f_i \equiv f$, $h_i \equiv h$, equation (4) passes into a meaningful difference equation for smooth coefficients, and at the same time, for these values of α and γ, the local error of the difference operator L_h does not exceed $O(h^2)$ (instead of $O(h)$ in the case $\alpha \neq \frac{2}{3}\gamma$). However, the condition $\alpha = \frac{2}{3}\gamma$ requires that the grid be almost square (the inequalities $h_i \leq h_{i+1} \leq \sqrt{2}\, h_i$ or $h_{i+1} \leq h_i \leq \sqrt{2}\, h_{i+1}$, $i = 1, 2, 3, 4$, must hold).

In the case of five ($\alpha = \gamma$) nodes, it is best to take $\alpha = \gamma = \frac{1}{2}$ (cf. equation (10) for $\delta = 0$ in [3]); in this case no restrictions are imposed on the steps h_i.

3°. For a regular hexagonal grid, the equation analogous to (4) can be written in the form

$$
\begin{aligned}
&\beta_1[(\lambda_1+\lambda_3)u_1+(\lambda_1+\lambda_2)u_2+(\lambda_2+\lambda_3)u_3]+\\
&\quad+\frac{1}{3}\left(\sqrt{3}\,\alpha-\beta_1-2\beta_2\right)[\lambda_1(u_4+u_5)+\lambda_2(u_6+u_7)+\lambda_3(u_8+u_9)]+\\
&\quad+\beta_2(\lambda_1u_{10}+\lambda_2u_{11}+\lambda_3u_{12})-\frac{1}{3}(\lambda_1+\lambda_2+\lambda_3)(4\beta_1-\beta_2+2\sqrt{3}\,\alpha)u_0=\\
&\quad=\frac{h^2}{3}(f_1+f_2+f_3)\left[3\alpha\beta_1+\frac{\sqrt{3}}{2}(3\alpha^2-\beta_1^2-2\beta_2^2)\right],
\end{aligned}
\tag{8}
$$

where u_i $(i=0,1,\dots,12)$ are the values of the function u, respectively, at the points

$$
(x,y),\quad (x+h,y),\quad \left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right),\quad \left(x-\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right),\quad \left(x+\frac{3}{2}h,y+\frac{\sqrt{3}}{2}h\right),
$$

$$
(x,y+\sqrt{3}h),\quad \left(x-\frac{3}{2}h,y+\frac{\sqrt{3}}{2}h\right),\quad \left(x-\frac{3}{2}h,y-\frac{\sqrt{3}}{2}h\right),\quad (x,y-\sqrt{3}h),
$$

$$
\left(x+\frac{3}{2}h,y-\frac{\sqrt{3}}{2}h\right),\quad (x+h,y+\sqrt{3}h),\quad (x-2h,y),\quad (x+h,y-\sqrt{3}h);
$$

λ_1 is the value of λ in the hexagon with side length h adjacent to the segments $(x,y),(x+h,y)$ and $(x,y),\left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)$; λ_2 is the value of λ in the hexagon adjacent to the segments $(x,y),\left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)$ and $(x,y),\left(x-\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)$, etc.; and the parameters α,β_1,β_2 (which have a quite definite geometric meaning) must satisfy the relations

$$
0\le\beta_1\le\sqrt{3},\quad 0\le\beta_2\le\frac{\sqrt{3}}{2},\quad 1\ge\alpha\ge\frac{\beta_1+2\beta_2}{\sqrt{3}}\quad(\alpha>0).
$$

In this case the local error does not exceed $O(h)$; if the parameters α,β_1,β_2 satisfy the equation

$$
2\beta_1-2\beta_2-\sqrt{3}\alpha=0,
$$

and the function U satisfies (2) and

$$
\left(\lambda\frac{\partial^2U}{\partial x\,\partial y}\right)_1=\left(\lambda\frac{\partial^2U}{\partial x\,\partial y}\right)_3,
$$

then it is $O(h^2)$, provided additionally that

$$
\beta_1=4\beta_2
$$

and U satisfies equation (1) and

$$\left(\lambda \frac{\partial^3 U}{\partial x^2\, \partial y}\right)_1 = \left(\lambda \frac{\partial^3 U}{\partial x^2\, \partial y}\right)_3 .$$

Thus the simplest four-point difference equation, obtained from (8) for

$$\beta_1 = \frac{\sqrt{3}}{2}, \quad \beta_2 = 0, \quad \alpha = \frac{1}{2},$$

has error $O(1)$. For $\lambda_i \equiv 1$, $f_i \equiv f$ (the case of smooth coefficients) and

$$\alpha = 6/11, \quad \beta_1 = 4\sqrt{3}/11, \quad \beta_2 = \sqrt{3}/11,$$

equation (8) becomes the seven-point equation considered in [4].

4°. In the case of a regular triangular grid, the use of additional nodes (over and above the usual seven) for constructing a homogeneous difference scheme is practically impossible. In this case, for continuous counting it is best to use the simplest seven-point formula of the type

$$\sum_{i=1}^{6} (\lambda_{i-1} + \lambda_i) u_i - 2 \sum_{i=1}^{6} \lambda_i u_0 = \frac{h^2}{2} \sum_{i=1}^{6} f_i \qquad (\lambda_0 = \lambda_6). \tag{9}$$

However, it should be noted that in the general case equation (9) has local error $O(1)$ (the density of discontinuities is large). In two particular cases

$$\lambda_i = \lambda_{i+1} = \lambda_{i+2} \neq \lambda_{i+3} = \lambda_{i+4} = \lambda_{i+5}$$

and

$$\lambda_i = \lambda_{i+2} = \lambda_{i+4} \neq \lambda_{i+1} = \lambda_{i+3} = \lambda_{i+5},$$

the local error of equation (9) does not exceed $O(h)$. Under

$$\lambda_i = \lambda_{i+1} = \lambda_{i+2} \neq \lambda_{i+3} = \lambda_{i+4} = \lambda_{i+5}$$

and certain other assumptions, the error is equal to $O(h^2)$.

5°. Analogous results can be obtained for the case when, instead of the first of conditions (2), the more general condition

$$[\mu U]_- = [\mu U]_+$$

is used (discontinuities are allowed not only in the derivatives, but also in the function itself). In this case the local error, in the sense indicated above, is not worsened in comparison with the previous case $\mu_i \equiv 1$.

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Note: Figure translations are in progress. See original paper for figures.