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Abstract

Full Text

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FINITE RING GROUPS

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1. In the present note we continue the study of ring groups begun in ^(1,2). It is devoted to questions connected with the notions of homomorphism, subgroup, and quotient group. We restrict ourselves only to finite ring groups, i.e., ring groups whose Hilbert space is finite-dimensional ⁽²⁾; the algebraic side of the question is completely clarified already in this simplest case. Moreover, in considering the infinite-dimensional case, additional difficulties arise connected with the invariant measure.
2. Let $\mathfrak{G} = (\mathfrak{M}, \Phi, +)$ be a finite ring group. The dimension of the ring \mathfrak{M} will be called the **order** of \mathfrak{G} . We shall normalize the invariant measure m so that $m(A) = \text{Sp}(A)$ ($A \in \mathfrak{M}$) (see (3) from ⁽²⁾). Let $\mathfrak{G}_1 = (\mathfrak{M}_1, \Phi_1, +)$ be another ring group. By definition, every homomorphism $f : A_1 \rightarrow A$ ($A_1 \in \mathfrak{M}_1$, $A \in \mathfrak{M}$) of the ring \mathfrak{M}_1 into \mathfrak{M} , satisfying the conditions: under f ,

$$A_1^+ \rightarrow A^+, \quad \Phi_1[A_1] \rightarrow \Phi[A], \quad (1)$$

defines a **homomorphism** \mathfrak{F} of the group \mathfrak{G} into \mathfrak{G}_1 . In connection with the last of relations (1), we note that $\Phi_1[A_1] \in \mathfrak{M}_1 \otimes \mathfrak{M}_1$, $\Phi[A] \in \mathfrak{M} \otimes \mathfrak{M}$, and that the homomorphism f is naturally extended to a homomorphism of the tensor product $\mathfrak{M}_1 \otimes \mathfrak{M}_1 \rightarrow \mathfrak{M} \otimes \mathfrak{M}$. In the case when f is an isomorphism of \mathfrak{M}_1 onto \mathfrak{M} , \mathfrak{F} will be called an **isomorphism** of the group \mathfrak{G} onto \mathfrak{G}_1 .

Let P be a central projector of the carrier \mathfrak{M} of the group \mathfrak{G} , satisfying the following two conditions:

$$P^+ = P, \quad \Phi[P] \supseteq P \otimes P. \quad (2)$$

The projector P defines the ring group $\mathfrak{P} = (P\mathfrak{M}, (P \otimes P)\Phi, +)$, which we shall call a **subgroup** of the ring group \mathfrak{G} . (The Hilbert space of \mathfrak{P} is $P\mathcal{H}$, where \mathcal{H} is the Hilbert space of \mathfrak{G} .) The projector P will be called the **principal projector** of the subgroup. The set of subgroups is partially ordered (in accordance with the ordering of their principal projectors). In particular, they all contain the **unit subgroup** \mathfrak{P}_ε . It is uniquely characterized by the fact that its principal projector is the unit of the group algebra $P_\varepsilon = \delta$ (see ⁽²⁾). The unit subgroup (as a ring group) is isomorphic to the unit group. The greatest subgroup belonging

to two given \mathfrak{P}_1 and \mathfrak{P}_2 is called their **intersection** $\mathfrak{P}_1 \cap \mathfrak{P}_2$. Its principal projector is equal to $P_1 P_2$. With each subgroup \mathfrak{P} one can associate a subring $\Omega \subseteq \mathfrak{M}$, formed by operators of the form $A * P$ ($A \in \mathfrak{M}$, P is the principal projector of \mathfrak{P}); we shall call it a (left) **homogeneous space**. (More precisely, the carrier of the homogeneous space. Similarly, a **right homogeneous space** can also be defined.) Lagrange's theorem carries over to ring groups.

Theorem 1. *The order of a subgroup of a ring group is a divisor of the order of the group.*

Let \mathfrak{G} be a Neumann subring of \mathfrak{M} (i.e., \mathfrak{G} contains the matrix I and, together with every matrix A , contains the adjoint matrix A^*), satisfying the conditions

$$\mathfrak{G}^+ = \mathfrak{G}, \quad \Phi[\mathfrak{G}] \subseteq \mathfrak{G} \otimes \mathfrak{G}. \quad (3)$$

Define a ring group \mathfrak{F} by putting $\mathfrak{F} = (\mathfrak{G}, \Phi, +)^*$. We shall call \mathfrak{F} a **factor group** of the group \mathfrak{G} . If the principal projector P of a subgroup \mathfrak{P} satisfies the condition $\Phi[P] = \Phi[P]$, then the ring Ω satisfies (3) and, consequently, defines a certain factor group. In this case we shall call the subgroup \mathfrak{P} **normal**. Unlike the case of ordinary groups, not all factor groups can be obtained in this way.

In Remark (1), for every ring group \mathfrak{G} a dual ring group $\widehat{\mathfrak{G}}$ and a Fourier transform are defined, establishing (in the finite-dimensional case) a one-to-one correspondence between the carriers \mathfrak{M} and $\widehat{\mathfrak{M}}$.

Theorem 2. *Let \mathfrak{G} and $\widehat{\mathfrak{G}}$ be mutually dual ring groups. Denote by $A \rightarrow \widehat{A}$ the corresponding Fourier transform. The Fourier transform maps the carrier of any subgroup (factor group) Γ of the group \mathfrak{G} onto the carrier of the dual group $\widehat{\Gamma}$, which is a factor group (subgroup) of $\widehat{\mathfrak{G}}$.*

Denote by P_0 the greatest central projector of the carrier \mathfrak{M} such that the ring $P_0 \mathfrak{M}$ belongs to the center of \mathfrak{M} .

Theorem 3. *P_0 is the principal projector of an ordinary subgroup \mathfrak{P}_0 (i.e., \mathfrak{P}_0 is isomorphic to an ordinary group). Every ordinary subgroup of the ring group \mathfrak{G} belongs to \mathfrak{P}_0 .*

In the case where \mathfrak{G} is an ordinary group, the notions introduced above coincide with the traditional ones.

Remark. R. A. Minlos showed that ring groups form a category and, independently of the author, introduced subgroups and factor groups. His definitions differ in form from those given above, but are equivalent to them. We shall not study the category of ring groups here. We give only one simple consequence of the definitions introduced above. With each subgroup \mathfrak{P} there is associated a natural homomorphism $\mathfrak{P} \rightarrow \mathfrak{G}$ (the corresponding $f : A \rightarrow PA$ ($A \in \mathfrak{M}$)). Similarly, with each factor group \mathfrak{F} there is associated a natural homomorphism $\mathfrak{G} \rightarrow \mathfrak{F}$ (f : the embedding $\mathfrak{G} \subseteq \mathfrak{M}$). Now let \mathfrak{F} be a homomorphism of the group \mathfrak{G} into \mathfrak{G}_1 . There exists a factor group \mathfrak{F} (of the group \mathfrak{G}) and a subgroup \mathfrak{P}_1

(of the group \mathfrak{G}_1) such that the homomorphism \mathfrak{F} can be represented in the form

$$\mathfrak{G} \xrightarrow{\mathfrak{F}'} \mathfrak{F} \xrightarrow[\mathfrak{F}''']{\sim} \mathfrak{P}_1 \xrightarrow{\mathfrak{F}''} \mathfrak{G}_1,$$

where \mathfrak{F}' and \mathfrak{F}''' are natural homomorphisms, and \mathfrak{F}'' is an isomorphism. In other words, \mathfrak{P}_1 and \mathfrak{F} are respectively the image and coimage of the homomorphism \mathfrak{F} .

- Let us consider some constructions leading to ring groups. Recall first of all that every group G can be interpreted in the following way as a ring group \mathfrak{G} . As the Hilbert space \mathfrak{H} one should take the space of functions $\varphi(x)$ on the group

$$\left(\|\varphi\|^2 = \sum_x |\varphi(x)|^2 \right).$$

Every function $f(x)$ is an operator in \mathfrak{H} : $\varphi(x) \rightarrow f(x)\varphi(x)$.

As \mathfrak{M} take the ring of all such operators and put

$$\Phi : f(y) \rightarrow f(xy), \quad f^+(x) = \bar{f}(x^{-1}).$$

Now let \mathfrak{G} be a ring group dual to the group G . It can be described in the following way*: \mathfrak{H} is the same as above, \mathfrak{M} is the ring,

* The ring \mathfrak{G} may be nonstandard; it satisfies all the other conditions in the definition of a ring group (1) \mathfrak{F} s

** This construction (in the case where G is an arbitrary unimodular group) is essentially due to Stinespring

generated by all left-shift operators $L_z : \varphi(x) \rightarrow \varphi(z^{-1}x)$. The mappings Φ and $+$ are defined on the operators L_z by the equalities

$$\Phi[L_z] = L_z \otimes L_z (: \varphi(x, y) \rightarrow \varphi(z^{-1}x, z^{-1}y)), \quad L_z^+ = L_z.$$

Let $\mathfrak{G}_1 = (\mathfrak{M}_1, \Phi_1, +)$ and $\mathfrak{G}_2 = (\mathfrak{M}_2, \Phi_2, +)$ be two ring groups. From them we construct a new ring group $\mathfrak{G} = (\mathfrak{M}, \Phi, +)$, which it is natural to call the **direct product** of the groups \mathfrak{G}_1 and \mathfrak{G}_2 : $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2 (= \mathfrak{G}_2 \times \mathfrak{G}_1)$. As \mathfrak{H} we take $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, where \mathfrak{H}_1 and \mathfrak{H}_2 are the Hilbert spaces of the groups \mathfrak{G}_1 and \mathfrak{G}_2 ,

respectively. Put $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$ and define the mappings Φ and $+$ on operators of the form $A \otimes B$ ($A \in \mathfrak{M}_1$, $B \in \mathfrak{M}_2$) by the equalities

$$\Phi[A \otimes B] = \Phi_1[A] \otimes \Phi_2[B], \quad (A \otimes B)^+ = A^+ \otimes B^+.$$

The operators $P_\varepsilon \otimes I$ and $I \otimes P_\varepsilon$ are the principal projectors of two normal subgroups, isomorphic respectively to \mathfrak{M}_1 and \mathfrak{M}_2 . The intersection of these subgroups is the unit subgroup.

Let us give one more, more complicated, analogous construction. Let G be an (ordinary) group and let G' be the group of its automorphisms. For each $\alpha \in G'$ denote by $\alpha(g)$ ($g \in G$) the corresponding automorphism. As \mathfrak{H} consider the Hilbert space of functions $\varphi(\gamma, k)$ ($\gamma \in G'$, $k \in G$,

$$\|\varphi\|^2 = \sum_{\gamma, k} |\varphi(\gamma, k)|^2),$$

and as \mathfrak{M} —the ring of operators

$$A : \varphi(\gamma, k) \rightarrow \sum_t a(\gamma, t) \varphi(\gamma, t^{-1}k).$$

A basis of this ring is formed by the operators $A_{\alpha, g}$:

$$\varphi(\gamma, k) \rightarrow \varphi(\alpha, g^{-1}k) \delta_\alpha(\gamma),$$

where δ_α is the δ -function concentrated at the point α ($\delta_\alpha(\gamma) = 0$ for $\gamma \neq \alpha$; $\delta_\alpha(\gamma) = 1$ for $\gamma = \alpha$). We define the mappings Φ and $+$ by setting

$$\Phi : A_{\alpha, g} \rightarrow \sum_{\pi \in G'} A_{\alpha \pi^{-1}, \pi(g)} \otimes A_{\pi, g}, \quad A_{\alpha, g}^+ = A_{\alpha^{-1}, \alpha(g)}. \quad (4)$$

A direct verification shows that all conditions α - γ^* of the definition of a ring group (1) are satisfied. Thus a ring group $\mathfrak{G} = (\mathfrak{M}, \Phi, +)$ has been constructed. From the definition of the operators $A_{\alpha, g}$ follow the relations

$$A_{\alpha, g} A_{\alpha', g'} = A_{\alpha, gg'} \delta_\alpha(\alpha'), \quad A_{\alpha, g}^* = A_{\alpha, g^{-1}},$$

showing that the operator $A_{\varepsilon, e}$ (ε, e are the units of the corresponding groups) is a central projector of the carrier \mathfrak{M} . Moreover, with the help of (4) we are convinced that the operator $A_{\varepsilon, e}$ satisfies conditions (2) and $\Phi[A_{\varepsilon, e}] = \Phi[A_{\varepsilon, e}]$. Thus, $A_{\varepsilon, e}$ is the principal projector of a normal subgroup. Its carrier $A_{\varepsilon, e} \mathfrak{M}$ is a linear combination of the operators $A_{\varepsilon, g}$. The correspondence $L_g \rightarrow A_{\varepsilon, g}$ defines an isomorphism of the carrier of the ring group \hat{G} (the dual group of G)

onto the carrier of the subgroup. In accordance with item 2 one can show that it defines an isomorphism of the subgroup onto \hat{G} . Thus, \mathfrak{G} contains a normal subgroup isomorphic to \hat{G} . An analogous argument shows that \mathfrak{G} contains a subgroup isomorphic to G' (its principal projector is proportional to $\sum_{\gamma,k} A_{\gamma,k}$). The intersection of these subgroups is the unit subgroup. What has been said above shows that the ring group \mathfrak{G} should be regarded as an analogue of the semidirect product of the ring group \hat{G} and the group of its automorphisms G' . This is also confirmed by the fact that in the case when all automorphisms $\alpha(g)$ are identical, \mathfrak{G} is the direct product $G' \times \hat{G}$.

4. In this section some questions of the representation theory of ring groups will be considered, connected mainly with the concepts of subgroup and factor group.

According to (2), the operator $P_\varepsilon = \delta$ is a minimal central projector of the ring \mathfrak{M} . Therefore, for every $A \in \mathfrak{M}$ the equality

* We take this opportunity to note that condition d) in (1) is superfluous; it follows from a)–c).

$AP(\varepsilon) = A(\varepsilon)P_\varepsilon$. The number $A(\varepsilon)$ may be interpreted as “the value of the function A at the point ε .” Let now $D(A)$ ($A \in \mathfrak{M}$; since the group is finite, $\mathfrak{M} = L_1(\mathfrak{M})$) be a representation of the ring group \mathfrak{G} ; N the order of this representation, and χ its character². In accordance with (5) of² and Theorem 2 of², we have

$$\begin{aligned} N = \text{Sp} D(P_\varepsilon) &= \sum D_{ii}(P_\varepsilon) = \sum (D(P_\varepsilon)\xi_i, \xi_i) = \sum m(P_\varepsilon Z_{ii}) = \\ &= \sum \text{Sp}(P_\varepsilon Z_{ii}) = \text{Sp}(P_\varepsilon \chi) = \chi'(\varepsilon) \text{Sp}(P_\varepsilon) = \chi(\varepsilon). \end{aligned}$$

The order of the representation D is equal to $\chi(\varepsilon)$.

The group algebra of a subgroup \mathfrak{P} is a subalgebra of the group algebra of the group. Therefore each representation $D(A)$ of the group may be regarded as a representation of the subgroup \mathfrak{P} . In doing so one must restrict oneself to the operators A belonging to the carrier $P\mathfrak{M}$ of the subgroup. Let Z_{ij} be the coefficients of the representation $D(A)$. According to (5) of², for all $A \in P\mathfrak{M}$ we have

$$(D(A)\xi_j, \xi_i) = m(AZ_{ij}) = \text{Sp}(AZ_{ij}) = \text{Sp}(APZ_{ij}).$$

This equality shows that the coefficients of the representation $D(A)$, regarded as a representation of the subgroup \mathfrak{P} , are the operators PZ_{ij} , where P is the principal projector of the subgroup. In particular, the character of this representation is the operator $P\chi$ (χ is the character of the representation $D(A)$ as a representation of the whole group).

Let us now consider the relations between representations and factor groups of a ring group. Let \mathfrak{F} be a factor group of the ring group \mathfrak{G} , $\mathfrak{S} \subseteq \mathfrak{M}$ the carrier of \mathfrak{F} , and $D(A)$ ($A \in \mathfrak{S}$) a representation of \mathfrak{F} . Denote by $Z_{ij} \in \mathfrak{S}$ the coefficients of the representation $D(A)$. It follows from conditions (6) of ² that the operators Z_{ij} also determine a representation of the whole ring group \mathfrak{G} . We shall call it a **representation of the factor group, considered on the whole group**. It has the same coefficients as the representation $D(A)$ of the factor group. Conversely, if $D(A)$ is a representation of the ring group with coefficients Z_{ij} , and there exists a factor group \mathfrak{F} such that all the operators Z_{ij} belong to its carrier, then $D(A)$ is described as a certain representation of the factor group \mathfrak{F} , considered on the whole group. We shall call a representation $D(A)$ of the ring group \mathfrak{G} **faithful** if it cannot be regarded as a representation of some factor group (different from \mathfrak{G}).

In the case of ordinary groups, the group itself is immediately reconstructed from a faithful representation of the group. An analogous situation also holds for ring groups, but the construction of the original group is more complicated. Let \mathfrak{G} be a ring group. We suppose that only the carrier \mathfrak{M} and some faithful representation $D(A)$ of the group \mathfrak{G} are known. Choose in the representation space a fixed orthonormal basis $\{\xi_i\}$. The equalities

$$D(A)_{ij} = (D(A)\xi_j, \xi_i) = \text{Sp}(AZ_{ij}) \quad (A \in \mathfrak{M})$$

make it possible to determine uniquely the operators $Z_{ij} \in \mathfrak{M}$, which are the coefficients of the representation $D(A)$. Denote by $\mathfrak{S} \subseteq \mathfrak{M}$ the $*$ -ring generated by the operators Z_{ij} . The equalities (6) of ² make it possible to determine uniquely on \mathfrak{S} the mappings Φ and $+$ of the group \mathfrak{G} (it should be borne in mind that Φ is an isomorphism). Moreover, for the ring \mathfrak{S} the conditions (3) turn out to be satisfied. Since the representation $D(A)$ is faithful, this means that $\mathfrak{S} = \mathfrak{M}$. Thus, the mappings Φ and $+$ on \mathfrak{M} have been obtained and, consequently, the ring group \mathfrak{G} is completely determined.

Theorem 4. *A ring group is uniquely determined by specifying its carrier \mathfrak{M} and some faithful representation $D(A)$.*

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Note: Figure translations are in progress. See original paper for figures.

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