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Abstract

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ON THE GROWTH OF HARMONIC FUNCTIONS OF THREE VARIABLES

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In note (1) a result was obtained that may be formulated as follows:

Theorem 1. Let $u(x, x_1, x_2)$ be a harmonic function of three variables in the half-cylinder

$$0 < x < \infty; \qquad (x_1, x_2) \subset D,$$

continuous together with its first partial derivatives up to its surface. Suppose that the boundary of the two-dimensional domain D satisfies Lyapunov' s conditions*. Denote by σ the width of the smallest layer between planes parallel to the x-axis in which the given half-cylinder can be placed.

If on the lateral surface of the half-cylinder the function $u(x, x_1, x_2)$ satisfies the conditions

$$|u(x, x_1, x_2)| + \left|\frac{\partial}{\partial n} u(x, x_1, x_2)\right| = O(x^{-\alpha}), \qquad x \to \infty;\ \alpha > 0,$$

and inside the half-cylinder

$$\iint_D |u(x, x_1, x_2)|\, dx_1\, dx_2 < C \exp\exp\frac{\pi}{\sigma + \eta} x, \qquad \eta > 0,$$

then

$$|u(x, x_1, x_2)| = O(x^{-\alpha}), \qquad x \to \infty; \quad (x_1, x_2) \subset \overline{D}.$$

The following problems naturally arise:

1) To extend the obtained result to arbitrary three-dimensional domains lying inside some layer or, at least, inside some rectangular half-cylinder.

2) To remove the smoothness requirements on the surface, which are most likely connected only with the method of proof.

Both problems would best be solved by proving a statement analogous to the principle of extension of the domain in the theory of harmonic measure.

In the present article, by means of extremely simple tools, the first of the posed problems is partially solved.

Consider an infinite three-dimensional domain V, which can be placed in a rectangular half-cylinder with generators parallel to one of the coordinate axes (we shall denote this axis by Ox). Let σ be the least width of all such half-cylinders. Suppose, moreover, that the surface S of the domain V satisfies Lyapunov' s conditions with constants fixed for the whole surface. Then the following assertion is valid.

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* For Lyapunov' s conditions see, for example, (2).

Theorem 2. Let $u(x, x_1, x_2)$ be a harmonic function of three variables, continuous together with its first partial derivatives inside the domain V and up to its boundary. If on the surface S of this domain

$$|u(P)| + \left|\frac{\partial}{\partial n} u(P)\right| = O(x^{-\alpha}), \qquad x \to \infty;\ \alpha > 0;\ P \subset S,$$

and for any section $D(x)$ of the domain V by the plane $x = \text{const}$

$$\iint_{D(x)} |u(x, x_1, x_2)|\, dx_1\, dx_2 < C \exp\exp \frac{\pi}{\sigma + \eta} x, \qquad \eta > 0,$$

then, uniformly inside V,

$$|u(P)| = O(x^{-\alpha}), \qquad P \to \infty.$$

Proof of Theorem 2. For the given function $u(x, x_1, x_2)$ construct the function

$$u_1(P) = \frac{1}{4\pi} \iint_S \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS$$

for any point $P(x, x_1, x_2)$ lying outside the domain V.

The function $u_1(P)$ is harmonic outside V. If $P \to \infty$ in any direction not parallel to the axis Ox, then $u_1(P) \to 0$.

We shall show that the function $u_1(P)$ can be harmonically continued to the whole space. Denote by S_0^R and S_R^∞ the surfaces of the domains V_0^R and V_R^∞, obtained by cutting the domain V by the plane $x = R$. The surfaces S_0^R and S_R^∞ have a common part D_R, and on D_R the directions of the outer normals to these surfaces are opposite. Since the point P lies outside V, by Green' s formula,

$$\iint_{S_0^R} \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS = 0,$$

whence

$$u_1(P) = \frac{1}{4\pi} \iint_{S_R^\infty} \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS$$

(the integrals over D_R cancel each other).

The last formula defines a function harmonic outside V_R^∞, which outside $V_\infty^0 \equiv V$ coincides with $u_1(P)$. In view of the arbitrariness of R, the continuation is carried out to the whole space.

Now, with the aid of Theorem 1, we shall show that $u_1(P) \equiv 0$. For this purpose enclose the domain V in a rectangular half-cylinder of width $\sigma + \eta_1$, where $\eta > \eta_1 > 0$, so that the distance between the surface S and the surface of the half-cylinder is positive. Then on the surface of the half-cylinder

$$u_1(P) = \frac{1}{4\pi} \iint_S \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS = O(x^{-\alpha}), \qquad x \to \infty,$$

and, obviously, the same estimate is valid for $\dfrac{\partial}{\partial n} u_1(P)$. Further, from the condition of the theorem being proved and from the boundedness of $u_1(P)$ outside the domain V, it follows that inside the half-cylinder

$$\iint_G |u_1(x, x_1, x_2)| \, dx_1 \, dx_2 < C \exp\exp \frac{\pi}{\sigma + \eta} x$$

(here D is the rectangle lying in the base). Thus, for a rectangular half-cylinder of width $\sigma + \eta_1$, all the conditions of Theorem 1 are satisfied (for an unqualified reference to Theorem 1 one must also "round off" the dihedral angles). This gives

$$|u_1(P)| = O(x^{-\alpha}), \qquad x \to \infty; \quad P \subset V.$$

Hence the function $u_1(P)$, harmonic in all space and bounded outside the domain V, turns out to be bounded inside it as well. Moreover,

$$\lim_{P \to \infty} u_1(P) = 0.$$

By Liouville' s theorem it follows that $u_1(P) \equiv 0$.

Let now $P(x, x_1, x_2)$ be an arbitrary point inside the domain V and let $R > x$. According to the well-known formula for expressing the values of a harmonic function inside a domain in terms of the values of this function and of its normal derivative on the boundary of the domain, we have

$$u(P) = \frac{1}{4\pi} \iint_{S_0^R} \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS.$$

But by what was proved above,

$$\frac{1}{4\pi} \iint_{S_R^\infty} \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS = u_1(P) = 0.$$

The last two relations give the formula

$$u(P) = \frac{1}{4\pi} \iint_{S} \left\{ \frac{\partial u}{\partial n} \frac{1}{r} - u \frac{\partial (1/r)}{\partial n} \right\} dS,$$

and together with it the assertion of the theorem.

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CITED LITERATURE

[1] I. S. Arshon, M. A. Evgrafov, DAN, **142**, No. 4 (1962).
[2] V. I. Smirnov, *A Course of Higher Mathematics*, 4, 2nd ed., 1951.

Note: Figure translations are in progress. See original paper for figures.