Reports of the Academy of Sciences of the USSR

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1962

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1962, Volume 142, No. 6

MATHEMATICS

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ON THE THEORY OF BOUNDARY-VALUE PROBLEMS WITH BOUNDARY CONDITIONS CONTAINING PARAMETERS

(Presented by Academician S. L. Sobolev on 20 VII 1961)

We shall regard the functions W_p^ν as defined in such domains D (the space $x_1, \dots, x_n$) with boundary S, for which the theory of these functional spaces applies (1–8,16,18).

Assuming $S = \sum_{s=1}^{n} S_{n-s}$, we divide it into two parts S^3 and S^c:

$$S = S^c + S^3 = \sum_{s=1}^{n} S^c_{n-s} + \sum_{s=1}^{n} S^3_{n-s}.$$

Then $W_p^\nu(S^3, 0)$ is the set of functions from W_p^ν, each of which, together with its derivatives up to order $\nu - [s/p] - 1$ inclusive, vanishes in the mean, with certain exponents (1,5), on the manifolds S^3_{n-s}. $W_p^\nu(S^3, 0)$ is closed in W_p^ν (5). $W_p^\nu(0) = W(S^3, 0)$ for $S^c_{n-s} = 0$; $W_p^\nu(0) \in W_p^\nu(S^3, 0)$. I shall give the formulation of the principal boundary-value problem for a certain special type of the equations and functionals under consideration. At the same time the results of the work also extend to functionals of a more general form, for example as in the papers (7,8).

Basic problem I. In W_p^ν, find a function u satisfying the variational equation

$$\int_D \cdots \int \left(\sum_{l=0}^{\nu} \sum_{\sum \alpha = l} \frac{\partial F_\nu^p(u)}{\partial u^l_{\alpha_1,\dots,\alpha_n}} \, \xi^l_{\alpha_1,\dots,\alpha_n} \right) dv = \int_D \cdots \int F_\nu^{p-1,1}(u, \xi)\, dv = 0, \quad (1)$$

where ξ is an arbitrary function from $W_p^\nu(0)$, under the following boundary conditions:

$$\int_D \cdots \int F_\nu^{p-1,1}(u,\eta)\,dv - \mu \sum_{s=1}^{n} \lambda_s \int_{S^c_{n-s}} \cdots \int \rho_s(x_1,\dots,x_n)\, F_{\nu_s}^{q_s-1,1}(u,\eta)\,dS_{n-s} = 0; \tag{2}$$

here η ranges over the entire set $W_p^\nu(S^3,0)$;

$$\sum_{s=1}^{n} \int_{S^c_{n-s}} \cdots \int \rho_s(x_1,\dots,x_n)\, F_{\nu_s}^{q_s}(u)\,dS_{n-s} = 1 \tag{3}$$

$\rho_s \geq 0$ on the manifolds S^c_{n-s}.

1) $F_\nu^p(u)$ is an admissible p-form in W_p^ν (^{7}); 2) F^{q_s} is an admissible form on S_{n-s} in W_ν^p of the connection S_{n-s} (^{7,8}); 3) μ and λ_s are constants; 4a) $0 \leq \nu_s \leq \nu - [s/p] - 1$; 4b) $1 < q_s \leq p(n-s)/(n - p(\nu - \nu_s))$ (q_s is any number > 1) if $n = p(\nu - \nu_s)$.

Remark. This problem is also posed and solved in the "weighted" classes $W^\nu_{p,b_1,\dots,b_s,\dots,b_n}$, which denotes the space whose elements have in the domain D generalized (in the sense of S. L. Sobolev) derivatives up to order ⋯

including v. Derivatives of order $v-1$ and lower are summable in D, and

$$\underbrace{\int^{n} \cdots \int}_{D} \prod_{s=1}^{n} r_{n-s}^{b_s} \sum_{\sum \alpha = v} \left| \frac{\partial^v u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p d\omega \tag{3_1}$$

has meaning. Here $r_{n-s}^{b_s}$ is the distance from points of the domain D to points of the boundary manifold S_{n-s}, raised to the power b_s; b_s is any number—the exponent of degeneration*. (For details on these spaces, see below.)

Some problems of type I have also been considered earlier ($^{8-10,17}$). In the present work, problems with linear and nonlinear equations of different orders, with boundary conditions mixed with nonlinearity, in domains with a degenerate contour, and also in "weighted" classes, are studied and solved for the first time. I shall explain the connection of the basic problem with boundary-value problems for differential equations by the simplest example. Let $n = p = 2$, $q < 4/b$, $q_1 < 2/b$, $b < 1$.

Proposition. There exists a number μ such that the differential equation**

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(r^b \frac{\partial u}{\partial x_i} \right) + f(x_1, x_2, u) = 0 \tag{1'}$$

has a solution $u \in W^1_{2,b}$, satisfying the following boundary conditions:

$$\int_{S_1^c} \left(r^b \frac{\partial u}{\partial n} - \mu\beta(S) u|u|^{q-2} \right) \eta \, dS = 0; \qquad \int_{S_2^c} r^b \frac{\partial u}{\partial n} \eta \, dS = 0; \qquad u|_{S^3} = 0. \tag{2'}$$

For any η from $W^1_{2,b}(S^3, 0)$ (generalized boundary condition)

$$\int_{S_0^c} r^b \frac{\partial u}{\partial n} \eta \, dS \to \int_{S_1^c} r^b \frac{\partial u}{\partial n} \eta \, dS$$

as $S_0 \to S_1^c$; S_0^c are boundaries of closed domains D_0 such that $D_0 \to D$. If $b = 0$ and the function u has a normal derivative on the contour S^c, then the boundary conditions $(2')$ take the form

$$\left.\frac{\partial u}{\partial n}\right|_{S_1^c} = \mu\beta(S) u|u|^{q-2}; \qquad \left.\frac{\partial u}{\partial n}\right|_{S_2^c} = 0; \qquad u|_{S^3} = 0;$$

here $S_1^c + S_2^c + S^3 = S$, the boundary of D. The problem can be expressed in terms of integral equations.

The following variational problem corresponds to Problem I. In $W^v_p(S^3, 0)$, find a function $\varphi = u$ realizing the minimum of the integral

$$\underbrace{\overset{n}{\int \cdots \int}}_{D} F^p_v(\varphi) \, dv \tag{4}$$

under condition (3).

Basic proposition A_1**.** *For arbitrary admissible forms of the kernel* F^p_v *(in* D*) and forms of the bond* $F^q_{\nu_1}$ *(on the manifolds* S_{n-s}*) there exists an infinite sequence of solutions* u_k *of the following recurrent variational problems. In* W^v_p*, find a function* $\varphi = u_k$ *realizing the minimum*

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* Instead of $r^{b_s}_{n-s}$, one may take more complicated distance functions.

** The equation may degenerate also on part of the boundary of the domain D; in this case the boundary conditions change correspondingly. $f(x_1, x_2, u)$ is a certain polynomial in u of degree $\leq q$ or a function of growth $|u|^q$. Analogous nonlinearities may also occur in the boundary conditions.

of the integral (1) under the additional conditions:

$$1) \quad \sum_{s=1}^{n} \left(\int_{S^c_{n-s}} \cdots \int \rho_s F^{q_s}_{\nu_s}(\varphi) \, dS_{n-s} \right)^{p/q_s} = 1;$$

$$2)\quad \sum_{s=1}^{n} \lambda_s \int_{S^c_{n-s}} \cdots \int \rho_s F_{\nu_s}^{q_s-1,1}(u_j,\varphi)\, dS_{n-s} = 0; \qquad j = 1, 2, \dots, k-1.$$

The functions u_k then satisfy equation (1) in the form

$$\int_D \cdots \int F_\nu^{p-1,1}(u_k,\eta_k)\, dv - \mu_k \sum_{s=1}^{n} \lambda_s \int_{S_{n-s}} \cdots \int F_{\nu_s}^{q_s-1,1}(u_k,\eta_k)\, dS_{n-s} = 0,$$

where

$$\int_D \cdots \int F_\nu^p(u_k)\, dv = \mu_k,$$

η_1 ranges over the whole set $W_p^\nu(S^3, 0)$.

In the case of quadratic kernel forms and connection functions, η_k range over the whole set $W_p^\nu(S^3, 0)$.

Proposition A_2**.** The functions u_k (eigenfunctions in A_1) are "orthogonal" and "normalized" as follows:

$$1)\quad \sum_{s=1}^{n} \lambda_s \int_{S^c_{n-s}} \cdots \int \rho_s F_{\nu_s}^{q_s-1,1}(u_j,u_i)\, dS_{n-s} = \begin{cases} 1, & j = i, \\ 0, & j < i; \end{cases}$$

$$2)\quad \int_D \cdots \int F_\nu^{p-1,1}(u_j,u_i)\, dv = \begin{cases} \mu_i, & j = i, \\ 0, & j < i. \end{cases}$$

Proposition A_3**.** The eigenvalues of equation (1) form an increasing sequence $\mu_k \to \infty$ as $k \to \infty$. The spectrum is countable and discrete.

Theorems A_1, A_2, and A_3 also extend to the corresponding problems considered in "weighted"classes. Here, instead of the subspaces $W_p^\nu(S^3, 0)$ and $W_p^\nu(S, 0)$, one uses the subspaces $W^\nu_{p,b_1\dots b_n}(S^3, 0)$ and $W^\nu_{p,b_1\dots b_n}(0)$, whose definition is obvious.

Let the kernel form in the principal problem be quadratic and of the same type as in (7).

Proposition A_4**.** There exists an infinite sequence of numbers $\mu_1, \mu_2, \dots, \mu_k, \dots$* such that the differential equation

$$L^\nu_{b_1,b_2,\dots,b_n}(u) + f\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{2m-2} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\right) = 0,$$

$$L^{\nu}_{b_1,b_2,\dots,b_n}(u) = \sum_{\sum\alpha=\nu} \frac{\partial^{\nu}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \left(\prod_{s=1}^{n} r_{n-s}^{b_s} A_{\alpha_1\dots\alpha_n}(x_1,\dots,x_n)\frac{\partial^{\nu} u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\right),$$

where f is the nonlinear part of (7), has a solution $u_k \in W^{\nu}_{2,b_1,b_2,\dots,b_n}$ satisfying generalized boundary conditions of the form (2), (3).

In the course of proving the propositions stated above, the properties of the functions W^{ν}_{p}, as well as additional integral inequality *. In the case of equations that "degenerate" on the boundary of the domain D or of a part of it, the corresponding "embedding" and "compactness" theorems were also applied. The "weighted" classes for the case $0 > b_s > -1$ were studied by the author already in (^{2}). The first systematic study of these classes for positive exponents in a certain class of domains (mainly direct and inverse "embedding" theorems) was carried out by L. D. Kudryavtsev in terms of the H-classes of S. M. Nikol' skii (^{12}), and subsequently by A. A. Vasharin (^{14}), P. I. Lizorkin (^{15}), and others (direct and inverse embedding theorems with integral-Hölder conditions on the boundary functions). The metric in $W^{\nu}_{p,b_1\dots b_n}$ can be introduced analogously to the way this was done for the classes W in (^{4,6}), with the factor $\prod_{s=1}^{n} r_{ns}^{b_s}$ for the highest derivatives, or in accordance with the problems under consideration. Then this space will be complete. In this connection the following holds:

Proposition A_5**.** a) *A set* $\rho(0,u) \le A$*, bounded in* $W^{\nu}_{p,b_1\dots b_n}$*, on the manifolds* $S_{n-s} \in \overline{D}$*, where* $s < pl - b_s$*, is compact in the spaces into which it is embedded:*

1) *in* $W^{\nu-k}_{q}$*, where*

$$q < q^{*} = \frac{p(n-s)}{n-kp+b_s}, \quad \text{if } n > kp - b_s \;\; **;$$

2) *in* $W^{\nu-k}_{q^{**}}$*, where* q^{**} *is any number* > 1*, when* $n = kp - b_s$*;*

3) *in*

$$C^{\nu-k} = C^{\nu-\left[\frac{s+b_s}{p}\right]-1}$$

when $n < kp - b$ $**$.

b) *The limiting functions* u *for a sequence* u_k*, selected according to the indicated compactness, belong to* $W^{\nu}_{p,b_1,\dots,b_n}$.

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* The numbers μ_k enter into the boundary conditions (see (2) and (2′)).

The convergence of the ν-th derivatives of the functions u_k to the corresponding derivatives of u is weak (in the sense of L_p) with a weight factor (see 3_1) ****.

The proofs of the stated theorems were carried out on the basis of a development of the methods of investigation used in ($^{1-8,16,17}$).

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Received
8 VII 1960

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* For example,

$$\int_D \cdots \int |u|^q \, d\nu \leq c \left(\int_D \cdots \int \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^p d\nu \right)^{q/p}; \qquad \int_D \cdots \int u \, d\omega = 0.$$

** The embedding and compactness theorems in the case W_{2,b_s}, $0 < b_s < 1$, the domain bounded, degeneration on all of S_{n-s} (a hyperplane), q not equal to q^* in the embedding theorems, were indicated by I. A. Solomesh (without the second part of theorem b).

*** In the embedding theorems: 1) $q = q^*$, for W_p^ν this was discovered for a number of cases by the author (^{5}), and then in the general case obtained by V. P. Il' in (^{18}); 2) the domain D may be unbounded under the same conditions as for W_p^ν.

**** This proposition is a development of the author' s result (^{4}).

Note: Figure translations are in progress. See original paper for figures.

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