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Abstract

Full Text

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ON THE UNIQUENESS OF GENERALIZED L_2 SOLUTIONS OF THE SIMPLEST PROBLEMS OF MATHEMATICAL PHYSICS

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Consider three simplest problems of mathematical physics:

1. The mixed problem for the wave equation

$$\Delta u - u_{tt} = f(x,t) \quad \text{in the cylinder } \Omega_l = g \times [0 \le t \le l]; \qquad (1)$$

$$u(x,0) = \varphi(x); \qquad u_t(x,0) = \psi(x); \qquad u\big|_{x\in\Gamma} = 0.$$

2. The mixed problem for the heat-conduction equation

$$\Delta u - u_t = f(x,t) \quad \text{in the cylinder } \Omega_l = g \times [0 \le t \le l]; \qquad (2)$$

$$u(x,0) = \varphi(x); \qquad u\big|_{x\in\Gamma} = 0.$$

3. The Dirichlet problem for the Poisson equation

$$\Delta u = f(x) \quad \text{in the domain } g,$$

$$u\big|_{x\in\Gamma} = \varphi. \qquad (3)$$

Here g is an arbitrary N-dimensional domain bounded by the surface Γ.

The aim of the present work is to prove the fact that each of problems 1, 2, and 3 can have only one generalized L_2 solution for a completely arbitrary bounded domain g.

Until now the uniqueness theorem for a generalized L_2 solution of problems 1, 2, and 3 had been proved only for the case when the domain g has a twice continuously differentiable boundary (see ([1–3])). Such a requirement of smoothness is all the more unnatural because the existence theorems for solutions of problems 1, 2, and 3 are proved for any bounded connected domain.

In view of the brevity of this article, we shall carry out the further arguments for the mixed problem for the wave equation (1); in doing so, we make essential use of the method proposed in ([4,5]).

Definition. A generalized L_2 solution of the mixed problem 1 is a function $u(x,t)$ that belongs to $L_2(\Omega_l)$ and satisfies the integral identity

$$\int_g \int_0^l \{u[\Phi_{0tt} - \Delta\Phi_0] - f\Phi_0\}\, dx\, dt + \int_g \left[\varphi \frac{\partial \Phi_0(x,0)}{\partial t} - \psi\Phi_0(x,0)\right] dx = 0 \quad (4)$$

for any function *

$$\Phi_0(x,t) \in \overset{0}{D}_2(\Omega_l) \cap W_2^{(2)}(\Omega_l). \quad (5)$$

Theorem. For an arbitrary bounded connected domain there can exist only one generalized L_2 solution of the mixed problem 1.

1°. Suppose that there exist two generalized L_2 solutions u_1 and u_2 of problem 1. Then the difference of these two solutions $u(x,t)$ will satisfy the identity

$$\int_g \int_0^l u\,[\Delta\Phi_0 - \Phi_{0tt}]\, dx\, dt = 0. \quad (4^*)$$

* For the definition of the indicated classes see, for example, ([6]), p. 39.

and will be a generalized solution from L_2 of the problem

$$\Delta u - u_{tt} = 0 \quad \text{in the cylinder } \Omega_l; \quad (1^*)$$

$$u(x,0) = 0; \qquad u_t(x,0) = 0; \qquad u|_\Gamma = 0.$$

First of all we shall prove the following

Auxiliary assertion. If $u(x,t)$ satisfies (4^*) for a function $\Phi_0(x,t)$ of class (5), then $u(x,t)$ must also satisfy (4^*) for functions $\Phi(x,t)$ of the form

$$\Phi(x,t) = v(x) \cdot \omega(t), \quad (5^*)$$

where $v(x)$ is one of the generalized* eigenfunctions of the problem

$$\Delta u + \lambda u = 0 \quad \text{in the domain } g;$$

$$u|_\Gamma = 0,$$

and $\omega(t)$ is a function twice continuously differentiable on the entire infinite line, equal to zero for $t \geq t_0$, where $t_0 < l$.

This function belongs to the class $\Phi(x,t) \in \overset{\circ}{D}_2(\Omega_l) \cap W_2^{(2)}(\Omega_l')$, where $\Omega_l' = g' \times [0 \leq t \leq l]$ and g' is a strictly interior subdomain of the domain g (*); moreover,

$$|\Delta\Phi - \Phi_{tt}| = -v(x)[\omega''(t) + \lambda\omega(t)] \in L_2(\Omega_l),$$

therefore the integral appearing on the left-hand side of (4*) exists.

Approximate the domain g by a sequence of domains g_m with boundaries $\Gamma_m \in A^{(2,\mu)}$ such that all $(g_m + \Gamma_m) \subset g$, and, whatever the closed set $g' \subset g$, all domains $g_m \supset g'$, starting with some number m.

To prove (4*) it is enough to show that

$$\lim_{m\to\infty} \int_{g_m} \int_0^l u[\Delta\Phi - \Phi_{tt}]\,dx\,dt = 0, \tag{6}$$

i.e. it is enough to prove that for every $\varepsilon > 0$ there is such an $m(\varepsilon)$ that

$$\left| \int_{g_m} \int_0^l u[\Delta\Phi - \Phi_{tt}]\,dx\,dt \right| < \varepsilon \quad \text{for } m \geq m(\varepsilon).$$

In the domain $(g_m + \Gamma_m)$ consider the following Dirichlet problem:

$$\Delta v_m = -\lambda v \quad \text{in the domain } g_m;$$

$$v_m|_{\Gamma_m} = 0.$$

Let v_m be the generalized solution of this problem.

Next, consider the function $w_m = v - v_m$, which is a generalized solution of the problem

$$\Delta w_m = 0 \quad \text{in the domain } g_m;$$

$$(w_m - v) \in \overset{\circ}{D}(g_m), \tag{7}$$

and show that

$$\|w_m\|_{W_2^{(1)}(g_m+\Gamma_m)} = \|v - v_m\|_{W_2^{(1)}(g_m+\Gamma_m)} < \varepsilon_1, \quad \text{where } \varepsilon_1 \to 0 \text{ as } m \to \infty. \tag{8}$$

The eigenfunction $v \in \overset{\circ}{D}(g)$, therefore there exists a sequence $\{\chi_q\}$ of functions continuously differentiable in g and vanishing in the boundary strip of the domain g, converging in the norm $W_2^{(1)}(g)$ to v, i.e. for any $\varepsilon_2 > 0$ there is such a $q(\varepsilon_2)$ that for $\varphi_0 = v - \chi_q$

$$\|\varphi_0\|_{W_2^{(1)}(g)} = \|v - \chi\|_{W_2^{(1)}(g)} < \varepsilon_2. \tag{9}$$

* For the definition of generalized eigenfunctions see ([6]), p. 44.

From the convergence of g_m to g it follows that, for $m \geq M(\varepsilon_2)$, $\chi_q \in \overset{\circ}{D}(g_m)$; hence, for $m \geq M(\varepsilon_2)$, problem (7) has become equivalent to the problem

$$\Delta w_m = 0 \quad \text{in the domain } g_m; \tag{7*}$$

$$w_m - \varphi_0 \in \overset{\circ}{D}(g_m).$$

Since φ_0 is an admissible function for problem (7^*), the solution of problem (7), or (7^*), $w_m = v - v_m$, by the definition of a generalized solution, satisfies the inequalities

$$E_{g_m}(w_m) \leq E_{g_m}(\varphi_0) \leq E_g(\varphi_0) \leq \text{const } \|\varphi_0\|_{W_2^{(1)}(g)},$$

where

$$E_{g_0}(z) = \int_{g_0} \left[\sum_{i,j=1}^{N} \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} \right] dx.$$

Thus, from (9) it follows that

$$\|\omega\|_{W_2^{(1)}(g_m + \Gamma_m)} < \varepsilon_1.$$

Introduce the notation $\Phi_m = v_m(x)\omega(t)$ and note that, in order to prove (6), it is sufficient that for any $\varepsilon_3 > 0$ there exist a number $m(\varepsilon_3)$ such that

$$\left| \int_{g_m} \int_0^l u \left[\Delta\Phi_m - (\Phi_m)_{tt} \right] dx\, dt \right| < \varepsilon_3 \quad \text{for } m \geq m(\varepsilon_3), \tag{10}$$

since, on the basis of (8),

$$|[\Delta\Phi - \Phi_{tt}] - [\Delta\Phi_m - (\Phi_m)_{tt}]| < \text{const}\cdot\varepsilon_1.$$

The function $v_m \in \overset{\circ}{D}(g_m) \cap W_2^{(2)}(g_m)$; therefore there exists a sequence $\{\tilde{v}_p\}$ of functions twice continuously differentiable inside g_m and equal to zero in a boundary strip, converging in the norm $W_2^{(2)}(g_m)$ to v_m. Extend the functions by zero to the domain $g - g_m$. To prove (10) it is sufficient to show that

$$\left|\int_{g_m}\int_0^l u\,[\Delta\widetilde{\Phi}_p - (\widetilde{\Phi}_p)_{tt}]\,dx\,dt\right| < \varepsilon_4, \quad \text{where } \varepsilon_4 \to 0 \text{ as } p \to \infty, \quad \widetilde{\Phi}_p = \tilde{v}_p\omega.$$

But $\tilde{v}_p \in W_2^{(2)}(g)$, and consequently $\widetilde{\Phi}_p$ also belongs to class (5), and integral (11) is simply equal to zero.

Thus we have fully proved the auxiliary assertion formulated above.

2°. Relying on the proved auxiliary assertion, it is easy to show that $u(x,t) = 0$ almost everywhere in the cylinder Ω_l. For this one must repeat the arguments contained, for example, in article ([4]) on pp. 120 and 121.

Remark 1. Analogous theorems are valid for problems 2 and 3.

Remark 2. The obtained results and the proposed method carry over, without any changes, to the case when in problems 1, 2, and 3, instead of the operator Δu, one takes an arbitrary self-adjoint elliptic operator

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u,$$

whose coefficients ensure that the generalized eigenfunctions of the problem

$$Lu + \lambda u = 0 \quad \text{in the domain } g;$$

$$u|_\Gamma = 0$$

belong to the class $W_2^{(2)}$ in an arbitrary strictly interior subdomain g' of the domain g.

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Note: Figure translations are in progress. See original paper for figures.