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## Abstract

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## PHYSICS

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# A MODEL OF SCALAR FIELD THEORY IN QUANTIZED SPACE-TIME

*(Presented by Academician N. N. Bogolyubov on 23 VI 1962)*

§ 1. In the present paper, using the example of a simple scalar model, we study certain generalizations of the apparatus of quantum field theory that are possible in the formalism of quantized space-time ($^{1-5}$). Since in the new scheme the coordinates $x^n$ become noncommuting operators, all constructions are carried out in $p$-space, which in the present case is a space of constant curvature. Taking into account the argument of § 2 of paper ($^5$), we shall define the 4-momentum $p_m$ $(m = 0, 1, 2, 3)$ by means of relation (5) from ($^5$) for $\varepsilon = 1$.

§ 2. Let $\psi(p)$ and $\varphi(k)$ be given scalar fields describing particles with masses $m$ and $\mu$, respectively, the field $\psi(p)$ being complex and $\varphi(k)$ real. The latter means (see, for example, ($^6$)) that under complex conjugation $\psi(p)$ goes into $\psi^*(-p)$, and $\varphi(k)$ into $\varphi(-k)$. We shall assume that the usual formalism of free fields is carried over into the new scheme with a change in only one point: instead of the function $\delta(p-q)$, which appears in the relativistically covariant notation of the commutation relations, of "normal" and "chronological" pairings, it is necessary to use, in accordance with the new character of integration in $p$-space, the function $\delta(p(-)q)$, defined by relation (19) from ($^5$).

Let us further suppose that the interaction Lagrangian of the model under consideration in the $x$-representation of the ordinary theory has the form $\mathcal{L}(x) = g\psi^+(x)\psi(x)\varphi(x)$ ( "cross" denotes Hermitian conjugation). To construct the $S$-matrix in curved $p$-space we must generalize the quantity*

$$\widetilde{\mathcal{L}}(q) = \int e^{-iqx}\mathcal{L}(x)\,dx = \frac{g}{\sqrt{2\pi}}\,\psi^+(q) * \psi(q) * \varphi(q), \tag{1}$$

since $\int \mathcal{L}(x)\,dx = \widetilde{\mathcal{L}}(0)$. It turns out that, despite the nonassociativity of the new convolution operation (see (17) in ($^5$)), this generalization, taking into account the requirement $\widetilde{\mathcal{L}}^+(q) = \widetilde{\mathcal{L}}(-q)$, is carried out uniquely**:

$$\widetilde{\mathcal{L}}(q) = \frac{g}{\sqrt{2\pi}} \left[\psi^{+}(q) * \psi(q)\right] * \varphi(q) \tag{2}$$

(here the symbol $*$ denotes the convolution (17) from ($^5$)), whence

$$\widetilde{\mathcal{L}}(0) = \frac{g}{\sqrt{2\pi}} \int \psi^{+}(p)\, \psi(-(p(+)k))\, \varphi(k)\, d\Omega_p\, d\Omega_k. \tag{3}$$

If the field operators are given in the interaction representation, then the $S$-matrix of the given model, taking (3) into account, can formally be written in the following form:

$$S = T \exp\left\{ i \frac{g}{\sqrt{2\pi}} \int \psi^{+}(p) \psi(-(p(+)k)) \varphi(k)\, d\Omega_p\, d\Omega_k \right\}, \tag{4}$$

where the symbol $T$ means that the functional (4) is brought to normal form according to Wick' s theorem for $T$-products, i.e., using "chronological" pairings.

In expanding (4) in powers of $g$, we obtain a set of Feynman diagrams with a modified law of addition of 4-momenta at the vertices,* a new volume element, and a new region of integration. As examples we give several of the simplest diagrams (see Fig. 1) and the expressions corresponding to them (the straight line in Fig. 1 corresponds to charged particles, the wavy line to neutral ones):

**Fig. 1**

$$\begin{aligned}
&a \quad \frac{ig}{\sqrt{2\pi}}\, N\left[\psi^{+}(-p)\psi(p(-)k)\varphi(k)\right];\\
&b \quad \frac{ig^2}{8\pi^2}\, \Delta^c(k)\, N\left[\psi^{+}(-p)\psi(p(-)k)\psi^{+}(-p')\psi(p'(+)k)\right];\\
&v \quad \frac{g^2}{16\pi^3}\, N\left[\psi^{+}(-p)\psi(p)\right] \int_{\Omega} \Delta^c(p(-)k)\Delta^c(k)\, d\Omega_k; \qquad (5)\\
&g \quad \frac{g^3}{6(2\pi)^{9/2}} \int d\Omega_k\, N\left[\psi^{+}(-(p(-)k(+)k'(+)k))\varphi(k')\psi(p)\right] \times\\
&\qquad\qquad \times D^c(p(-)k)\Delta^c(k)D^c(p(-)k(+)k').
\end{aligned}$$

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* To write $\widetilde{\mathcal{L}}(q)$ the convolution symbol is used; the Fourier images of the fields $\psi(x)$ and $\varphi(x)$ are defined as in ($^6$).

** For self-interacting scalar fields, for example the Heisenberg–Tirring field ($^7$), the generalization of $\widetilde{\mathcal{L}}(q)$ is especially simple:

$$\widetilde{\mathcal{L}}(q) = \frac{g}{\sqrt{2\pi}} \varphi(q) * \varphi(q) * \varphi(q) = \frac{g}{\sqrt{2\pi}} \int \varphi(-(p(-)q))\varphi(k)\varphi(-(k(-)p))\, d\Omega_p\, d\Omega_k.$$

($N$ is the symbol of normal ordering.)

§ 3. We shall begin the investigation of the convergence of the integrals contained in the expansion of the matrix (4) by considering diagram $v$, which in the ordinary theory corresponds to the logarithmically divergent integral

$$\Sigma_2(p^2) = \frac{1}{i} \int D^c(p-k)\Delta^c(k)\, d^4k. \tag{6}$$

As is known, (6) can also be represented in the form

$$\Sigma_2(p^2) = \lim_{L\to\infty} \int_0^1 d\alpha \int_{k_\nu^2 \le L^2} \frac{dk_4\, dk}{(M + k_4^2 - i\varepsilon)^2} \quad (\nu = 1, 2, 3, 4), \tag{7}$$

where $M = (1-\alpha)\mu^2 + \alpha m^2 - \alpha(1-\alpha)p^2 + k^2 > 0$ for $p^2 < (m+\mu)^2$, and the internal integration is carried out over Euclidean 4-space. In the $S$-matrix (4), according to (5), instead of (6) we shall have

$$\Sigma_2(p^2) = \frac{1}{i} \int_\Omega D^c(p(-)k)\Delta^c(k)\, d\Omega_k \tag{8}$$

or, in explicit form (taking account of (10) and (15) from (5)),

$$\Sigma_2(p^2) = \frac{1}{i} \int_{k^2 > -1} \frac{d^4k}{\sqrt{1+k^2}\,[1 + m^2 - (\sqrt{1+p^2}\sqrt{1+k^2} - pk)^2 - i\varepsilon](\mu^2 - k^2 - i\varepsilon)}. \tag{9}$$

For (9) one can find a parametric representation analogous to (7) *

$$\Sigma_2(p^2) = -\frac{1}{i} \int_0^1 \frac{d\alpha}{1 + 4\alpha(1-\alpha)p^2} \int_\Omega \frac{dk_0\, dk}{\sqrt{1+k^2}\,(M - k_0^2 - i\varepsilon)^2}, \tag{10}$$

where

$$M = \frac{(1-\alpha)\mu^2 + \alpha m^2}{\sqrt{1 + 4\alpha(1-\alpha)p^2}} + \frac{(1+k^2) - (1-k^2)\sqrt{1+4\alpha(1-\alpha)p^2}}{2\sqrt{1+4\alpha(1-\alpha)p^2}} > 0 \tag{11}$$

for $-1 \le p^2 < (m(+)\mu)^2 = (m\sqrt{1+\mu^2} + \mu\sqrt{1+m^2})^2$.

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* In this case the order of the summed 4-momenta, because of the single-valuedness of expression (3) for $\mathscr{F}(0)$ and the property $\delta(p(-)q) = \delta(q(-)p)$ of the function $\delta(p(-)q)$ entering into the "chronological" pairings, is completely definite and is in accordance with the rule given in (3).

However, in representation (10), unlike (7), the form of the region $\Omega$ and the analytic properties of the integrand do not allow one to rotate the integration path in the $k_0$-plane by $\pi/2$ and thus pass to integration over a curved $p$-space with positive definite metric (hereafter $E_4$), which is the analogue of Euclidean 4-space in (14). For this reason the integral (9)—(10), despite its "radial" convergence, turns out to be divergent with respect to the hyperbolic angular variables (cf. § 26 of ($^6$), where an analogous situation with divergence with respect to ordinary angles is discussed).

If the transition to integration over the space $E_4$ were possible, then for the integral under consideration we would have the explicitly convergent expression

$$\Sigma_2'(p^2) = \int_0^1 \frac{d\alpha}{1+4\alpha(1-\alpha)p^2} \int_{k_\nu^2 \le 1} \frac{dk_4\,dk}{\sqrt{1-k_\nu^2}\,(M+k_4^2-i\varepsilon)^2} \qquad (\nu = 1,2,3,4), \tag{12}$$

which is a real quantity for values of $p^2$ below the "threshold" $(m(+)\mu)$ (see (11)). The integral (12) is evidently completely analogous to (7). We can write these expressions in an even more similar form if in each of them we carry out analytic continuation in the variable $p_0$ from its real values to purely imaginary values ** $ip_4$ and perform the integration over $\alpha$. As a result we obtain:

$$\Sigma_2(-p_\nu^2) = \lim_{L\to\infty} \int_{k_\nu^2 \le L^2} \frac{dk}{[m+(p-k)_\nu^2](\mu^2+k_\nu^2)} \qquad (dk = dk_4\,dk); \tag{13}$$

$$\Sigma_2'(-p_\nu^2) = \int_{k_\nu^2 \le 1} \frac{d\Omega_k}{[m^2+(p(-)k)_\nu^2](\mu^2+k_\nu^2)}, \tag{14}$$

where $(p(-)k)_\nu^2 = 1-(\sqrt{1-p_\nu^2}\sqrt{1-k_\nu^2}+p_\nu k_\nu)^2$ is the square of the vector transformed by the displacement operation in $E_4$, and $d\Omega_k = dk/\sqrt{1-k_\nu^2}$ is the volume element of this space.

§ 4. Analysis shows that divergences with respect to hyperbolic angles also arise in higher orders of perturbation theory. Therefore the method given in § 3 for constructing the finite integral $\Sigma_2'$ (see (12) and (14)) should appropriately be extended to the case of an arbitrary diagram, i.e., we arrive at the conclusion that the scattering matrix in the scheme under consideration must be a generalization of this form of the ordinary $S$-matrix, in which all internal integrations are performed over Euclidean 4-space. With such an approach the role of the new $S$-matrix can be played by a functional *** (cf. ($^9$))

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* (10) is a relativistically invariant representation of $\Sigma_2(p^2)$, since the 4-vector $p_m$ enters the integrand only in the form $p^2$. Obviously, as $l \to 0$, (10) goes over into the Feynman representation (7).

** The possibility of such an analytic continuation for (7) is obvious, while for (12) it can be proved rigorously.

*** An analogous functional was considered by Yu. A. Golfand ($^8$).

$$S' = e^{\Delta+\Sigma} \exp\left[-\frac{ig}{\sqrt{2\pi}} \int \psi^+(p)\, \psi(-(p(+)k))\, \varphi(k)\, d\Omega_p d\Omega_k\right],$$

where

$$\Delta = \frac{1}{4\pi} \frac{d\Omega_k}{\mu^2 + k_\nu^2} \frac{\delta^2}{\delta\varphi(k)\delta\varphi(-k)}, \qquad \Sigma = \frac{1}{2\pi} \int d\Omega_p \frac{\delta}{\delta\psi(p)} \frac{1}{m^2 + p_\nu^2} \frac{\delta}{\delta\psi^+(-p)}. \tag{15}$$

Here all integrations are performed over the space $E_4$, and the functional derivatives are understood in the sense of (21) and (22) from (5). The "true" matrix elements are obtained by varying $S'$ with respect to the arguments $\psi, \psi^+, \varphi$, followed by setting these arguments equal to zero (cf. § 47 of (6)) and by analytic continuation of the type $p_4 \to -ip_0$ of the expressions obtained into the physical range of values of the 4-components of the external momenta.* For example, for the second-order mass diagram considered above we have:

$$\left.\frac{\delta^2 S_2'}{\delta\psi^+(-p)\delta\psi(p)}\right|_{\psi,\psi^+,\varphi=0} = -\frac{g^2}{16\pi^3}\, \delta(p(-)p')\, \Sigma_2'(-p_\nu^2);$$

$$\Sigma_2'(p_0^2 - \mathbf{p}^2) = \text{analytic continuation of } \Sigma_2'(-p_\nu^2),$$

where $S_2'$ denotes the second-order terms in (15). A characteristic feature of the integrals obtained in this way is the absence of divergences. The fulfillment of the unitarity condition for the first two orders in $g$ is obvious, while for higher approximations a special investigation is required.

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* In doing so, in order to handle correctly the singularities arising in the physical region, it is necessary to add $-i\varepsilon$ to the masses. The elements of the usual scattering matrix can evidently be obtained in an analogous way. The possibility of the analytic continuation $p_4 \to -ip_0$ in this case is proved in general form in (6).

*Note: Figure translations are in progress. See original paper for figures.*

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