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Abstract

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ON CARLEMAN’ S PROBLEM FOR THE CLASS OF GEVREY FUNCTIONS

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Let $\{m_n\}$ be some sequence of positive numbers. Denote by $C(m_n)$ the class of all infinitely differentiable functions $f(x)$ on the segment $[-1,1]$ for each of which there exists an $A > 0$ such that

$$|f^{(n)}(x)| \leq A^{n+1} m_n$$

for all $x \in [-1,1]$ and all $n = 0, 1, 2, \ldots$.

In the present paper the following problem is solved. For any $\alpha > 1$ and any sequence of complex numbers $\{v_n\}$ satisfying the condition $|v_n| \leq B^{n+1} n^{n\alpha}$ for some $B > 0$, the existence is proved of a function $f(x) \in C(n^{n\alpha})$ such that $f^{(n)}(0) = v_n$. Moreover, the function $f(x)$ is written explicitly.

Remark 1. This problem is analogous to the problem posed by Carleman (^{1}) for the case when the class $C(m_n)$ is quasianalytic. The class $C(n^{n\alpha})$ is not quasianalytic for any $\alpha > 1$.

Remark 2. In the work of B. S. Mityagin (^{4}) the fact of the existence of the above-mentioned function $f(x)$ is proved. In the present paper the function $f(x)$ is constructed explicitly in the form (1), which is an analogue of the Taylor series.

For the proof of the assertion formulated by us, consider the series

$$\sum_{k=0}^{\infty} \frac{v_k}{k!} a_k(x) x^k, \tag{1}$$

where the numbers v_k are the same as above, and the functions $a_k(x)$ are defined as follows.

Let $b_k(x)$ be the functions given by the equalities

$$b_k(x) = \begin{cases} 0, & \text{for } -1 \le x \le -\sigma_k, \\ \exp\left(-\dfrac{k\sigma_k^{4r}}{x^{2r}(\sigma_k + x)^{2r}}\right), & \text{for } -\sigma_k \le x \le 0, \\ 0, & \text{for } 0 \le x \le 1, \end{cases} \tag{2}$$

where r is any natural number such that $\dfrac{1}{2r} < \alpha - 1$, and $\sigma_k = D^{-1}k^{-(\alpha-1)}$ for some $D > 0$. Put

$$a_k(x) = \frac{\displaystyle\int_{-1}^{x} b_k(t)\,dt}{\displaystyle\int_{-1}^{1} b_k(t)\,dt} \tag{3}$$

for $-1 \le x \le 0$, and for $0 < x \le 1$ put $a_k(x) = a_k(-x)$.

We shall now prove that the series (1) converges, together with all its derivatives, uniformly on $[-1, 1]$, and that its sum $f(x)$ gives a solution of the problem, i.e. $f(x) \in C(n^{n\alpha})$, and $f^{(n)}(0) = v_n$. (The value of the number $D > 0$, entering into $b_k(x)$, will be specified later.) Let us estimate $\left|a_k^{(n)}(x)\right|$.

First of all, let us estimate from below the integral $\displaystyle\int_{-1}^{1} b_k(t)\,dt$. From (2) it is clear that $b_k(x)$ increases for $-\sigma_k < x < -\sigma_k/2$ and decreases for $-\sigma_k/2 < x < 0$, and, moreover, $b_k(-\sigma_k/2 - t) = b_k(-\sigma_k/2 + t)$ for $0 \le t \le \sigma_k/2$. Therefore,

$$\int_{-1}^{1} b_k(t)\,dt = \int_{-\sigma_k}^{0} b_k(t)\,dt > \frac{\sigma_k}{2}\, b_k\left(-\frac{\sigma_k}{4}\right)$$

or, by virtue of (2),

$$\int_{-\sigma_k}^{0} b_k(t)\,dt > \frac{\sigma_k}{2} \exp\left(-\frac{16^{2r}k}{3^{2r}}\right). \tag{4}$$

Let us estimate from above $\left|b_k^{(n)}(x)\right|$. We shall use the fact that the function $b_k(x)$ is analytic on the interval $(-\sigma_k, 0)$, and apply Cauchy' s formula to the derivative $b_k^{(n)}(x)$, taking as the contour of integration the circle with radius equal to $-hx$, and center at the point $x \in [-\sigma_k/2, 0)$. We obtain

$$b_k^{(n)}(x) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{b_k(x - hxe^{i\varphi})}{(-hxe^{i\varphi})^n}\, d\varphi.$$

Hence

$$\left|b_k^{(n)}(x)\right| \le \frac{n!}{2\pi}\,\frac{1}{h^n|x|^n}\int_0^{2\pi}\left|b_k(x-hxe^{i\varphi})\right|\,d\varphi.$$

On the other hand, after simple transformations we obtain

$$\left|b_k(x-hxe^{i\varphi})\right| = \exp\left(\frac{-k\sigma_k^{4r}\cos(2r\beta+2r\gamma)}{x^{2r}(1-2h\cos\varphi+h^2)^r\left[(\sigma_k+x-hx\cos\varphi)^2+h^2x^2\sin^2\varphi\right]^r}\right),$$

where β and γ are defined by the relations

$$\sin\beta = \frac{h\sin\varphi}{(1-2h\cos\varphi+h^2)^{1/2}},\qquad \sin\gamma = \frac{hx\sin\varphi}{\left[(\sigma_k+x-hx\cos\varphi)^2+h^2x^2\sin^2\varphi\right]^{1/2}}.$$

We see that if h is sufficiently small, then

$$\left|b_k(x-hxe^{i\varphi})\right| \le \exp\left(-\frac{L\sigma_k^{2r}k}{x^{2r}}\right),$$

where $L>0$ depends only on r. Consequently,

$$\left|b_k^{(n)}(x)\right| \le \frac{n!}{h^n|x|^n}\exp\left(-\frac{L\sigma_k^{2r}k}{x^{2r}}\right).$$

It is easy to show that

$$\max_{\sigma_k/2\le x\le 0}\left\{\frac{1}{|x|^n}\exp\left(-\frac{L\sigma_k^{2r}k}{x^{2r}}\right)\right\} \le \frac{1}{\sigma_k^n}\left(\frac{n}{k}\right)^{n/2r}\left(\frac{1}{2rL}\right)^{n/2r}e^{-n/2r}.$$

Thus, we have the following estimate for $\left|b_k^{(n)}(x)\right|$:

$$\left|b_k^{(n)}(x)\right| \le \frac{T^n n! n^{n/2r}}{\sigma_k^n k^{n/2r}} \le \frac{T^n D^n n^n n^{n/2r} k^{n(\alpha-1)}}{k^{n/2r}}. \tag{5}$$

Here $T=\max(T_1,1)$, where

$$T_1 = \frac{1}{h}\left(\frac{1}{2rL}\right)^{1/2r}e^{-1/2r}.$$

It is clear that T does not depend on n and k. Using inequalities (4) and (5), we arrive at the following estimate for $\left|a_k^{(n)}(x)\right|$:

$$\left|a_k^{(n)}(x)\right| \leq 2\exp\left(\frac{16^{2r}k}{3^{2r}}\right)T^nD^n\frac{n^n n^{(n-1)/2r}k^{n(\alpha-1)}}{k^{(n-1)/2r}}. \tag{6}$$

Differentiating the series (1) n times, we obtain a series with general term

$$w_k(x) = \frac{v_k}{k!}\sum_{i=0}^{n}\binom{n}{i}a_k^{(n-i)}(x)(x^k)^{(i)}.$$

On the basis of (6)

$$\begin{aligned}
|w_k(x)| &\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right)\sum_{i=0}^{n}\binom{n}{i}T^{\,n-i}D^{n-i}\frac{(n-i)^{n-i}(n-i)^{(n-i-1)/2r}}{k^{-(n-i)(\alpha-1)}k^{(n-i-1)/2r}}\,k^i\delta_k^{k-i} \leq \\
&\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right)\sum_{i=0}^{n}\binom{n}{i}T^{\,n-i}D^{n-i}\frac{(n-i)^{n-i}(n-i)^{(n-i-1)/2r}}{k^{-(n-i)(\alpha-1)}k^{(n-i-1)/2r}}\,k^iD^{-k+i}k^{-(k-i)(\alpha-1)} \leq \\
&\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}}{3^{2r}}\right)\sum_{i=0}^{n}\binom{n}{i}T^nD^nD^{-k}\frac{(n-i)^{n-1}(n-i)^{(n-i-1)/2r}k^ik^{-k(\alpha-1)}}{k^{-n(\alpha-1)}k^{(n-i-1)/2r}} \leq \\
&\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right)D^{-k}k^{-k(\alpha-1)}\sum_{i=0}^{n}\binom{n}{i}T^nD^n\frac{n^{n-i}n^{(n-i-1)/2r}k^i}{k^{-n(\alpha-1)}k^{(n-i-1)/2r}}.
\end{aligned}$$

For $k \leq n$ we shall have

$$\begin{aligned}
|w_k(x)| &\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right)D^{-k}k^{-k(\alpha-1)}\sum_{i=0}^{n}\binom{n}{i}T^nD^n\frac{n^{n-i}n^{(n-i-1)/2r}n^i}{n^{-n(\alpha-1)}n^{(n-i-1)/2r}} \leq \\
&\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right)D^{-k}k^{-k(\alpha-1)}\sum_{i=0}^{n}\binom{n}{i}T^nD^nn^{n\alpha}.
\end{aligned}$$

By Stirling' s formula and the inequality $|v_k| \leq B^{k+1}k^{k\alpha}$, we have

$$|w_k(x)| \leq 2^{\,n-k+1}T^nD^nn^{n\alpha}$$

for sufficiently large D. Therefore,

$$\sum_{k=0}^{n}|w_k(x)| \leq 2^{\,n+2}T^nD^nn^{n\alpha}.$$

Now let $k > n$. Then

$$|w_k(x)| \leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right) D^{-k}k^{-k(\alpha-1)}\sum_{i=0}^{n}\binom{n}{i}\frac{k^{n-i}k^{(n-i-1)/2r}k^{i}}{k^{-n(\alpha-1)}k^{(n-i-1)/2r}} \leq$$

$$\leq 2\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{2r}}\right) D^{-k}k^{-k(\alpha-1)}\sum_{i=0}^{n}\binom{n}{i}T^nD^nk^{n\alpha}.$$

Thus,

$$\sum_{k=n+1}^{\infty}|w_k(x)| \leq 2^{n+1}T^nD^n\sum_{k=n+1}^{\infty}\frac{|v_k|}{k!}\exp\left(\frac{16^{2r}k}{3^{r}}\right) D^{-k}k^{-k(\alpha-1)}k^{n\alpha}.$$

Applying Stirling' s formula once more, we obtain

$$\sum_{k=n+1}^{\infty}|w_k(x)| \leq 2^{2n+1}T^nD^nMn^{n\alpha},$$

where for M one may take, for example, the number $(2\alpha)^{\alpha}$, and for D the number $4B\exp\left(\frac{16^{2r}}{3^r}+2\right)$. As a result we have

$$\sum_{k=0}^{\infty}|w_k(x)| \leq 2^{n+2}T^nD^nn^{n\alpha}+2^{2n+1}T^nM^nD^nn^{n\alpha} \leq 8^{n+1}T^nD^nMn^{n\alpha} \leq A^{n+1}n^{n\alpha},$$

where $A = 8TDM$ and does not depend on n.

Thus the uniform convergence of the series (1) and the membership of its sum $f(x)$ in the class $C(n^{n\alpha})$ have been proved. The fact that $f^{(n)}(0) = v_n$ follows from the definition of $a_k(x)$.

As an application, let us find, for the space S_{β}^{α} (see (2)) with $\alpha > 1$, the general form of a linear continuous functional concentrated at a point. (This result was obtained by another method in the paper [3].)

Let $H(\varphi)$ be a linear functional concentrated at zero, and $\varphi \in S_{\beta}^{\alpha}$. From the preceding it is clear that the series

$$\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}a_k(x)x^k$$

converges in S_{β}^{α}. Hence

$$H(\varphi) = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} H(a_k(x)x^k). \tag{7}$$

Since $\varphi^{(k)}(0)$ may be any sequence of complex numbers satisfying the condition

$$|\varphi^{(k)}(0)| \leq B^{k+1} k^{k\alpha},$$

it follows that, for every $\varepsilon > 0$, starting from some k,

$$\left|H(a_k(x)x^k)\right| \leq \varepsilon^k k^{-k(\alpha-1)}.$$

Conversely, if the last condition is fulfilled, formula (7) defines a functional concentrated at the point $x = 0$.

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Note: Figure translations are in progress. See original paper for figures.

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