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Abstract

Full Text

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On Four-Point Schemes of Increased Accuracy

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It is widely held that, because of the low accuracy of the local approximation (of order h) of the Laplace operator by a difference operator written on a hexagonal grid with four points, such difference schemes are unlikely to be of great practical importance (see, for example, ([1])). To eliminate this shortcoming, in ([3]) seven-point schemes on a hexagonal grid were constructed. However, four-point schemes approximate the Laplace operator well in the weak sense, owing to the fact that the principal term of the approximation error has opposite signs at neighboring points ([2]); thanks to this it will be shown that the error between the approximate and a sufficiently smooth exact solution, obtained when four-point schemes are applied, will be of order $O(h^3)$, and moreover, for some schemes a set A_h^1 is indicated, forming an h-grid, on which the error will be $O(h^4)$. The difference equations obtained below can be obtained in two ways: 1) we seek good group approximation, so that a certain combination of four-point equations forms a difference equation that locally approximates the Laplace operator well; 2) into a system of equations, each equation of which locally approximates the Laplace operator well, we introduce new unknowns in such a way that each equation of the newly obtained system contains fewer unknowns than in the original one, and the matrix of the new system is symmetric and negative definite. Geometrically the new unknowns may be regarded as the approximate solution on a certain set B_h^1.

Let the function u be a solution of the equation

$$\Delta u = f, \tag{1}$$

defined inside a domain Ω with boundary S in the plane (x_1, x_2). Construct in the plane (x_1, x_2) a hexagonal grid D_h with mesh size h. We divide the set of nodes of the grid D_h into two subsets: A_h and B_h. Take from D_h some point x_0 and suppose that $x_0 \in A_h$; assign the three points of D_h adjacent to it to the set B_h, the points adjacent to them again to the set A_h, and so on. Thus the points adjacent to points of the set A_h (respectively B_h) belong to the set B_h (respectively A_h). The fact that all points $x \in A_h$ which are adjacent to points that are adjacent to some point $x_0 \in A_h$ and are different from x_0, are located at the same distance from the point x_0 and form a regular lattice will allow us to increase the accuracy of the difference schemes. From the set A_h, with respect to the domain Ω, we single out the subsets of interior points A_h^1

and boundary points A_h^2. A point $x \in A_h^1$ if $x \in A_h$ and the regular hexagon with center at the point x and with vertices at the points of the set A_h adjacent to x is entirely contained in Ω. The remaining points of the set A_h lying in Ω are assigned to A_h^2. By B_h^1 denote the subset of B_h such that each of its points is surrounded by three points of the set $A_h^1 + A_h^2$, among ···

of which there is at least one point of the set A_h^1. Take any point $x \in A_h^1$ as the origin of coordinates, and denote it by the index 0; let the points 1, 2, 3 of the set B_h^1, neighboring the point x, have respectively the coordinates $(0, h)$, $\left(-\frac{\sqrt{3}}{2}h, -\frac{h}{2}\right)$, $\left(\frac{\sqrt{3}}{2}h, -\frac{h}{2}\right)$, and let the points 4, 5, 6, 7, 8, 9 of the set $A_h^1 + A_h^2$ be neighboring to the points 1, 2, 3.

Then, if u is a sufficiently smooth function, then

$$\Delta_h'' u_0 \equiv \frac{2}{9h^2}\left(\sum_{i=4}^{9} u_i - 6u_0\right) = \Delta u_0 + \frac{3h^2}{16}\Delta^2 u_0 + O(h^4), \tag{2}$$

$$\Delta_h' u_i = \frac{4}{3h^2}\left(\left(\sum u\right)_i - 3u_i\right) =$$

$$= \Delta u_i + \frac{h^2}{16}\Delta^2 u_i + \frac{\delta h}{6}\left(u_{x_3^3} - 3u_{x_1^2 x_2}\right)_i + \delta h M_5(u)_i + O(h^4), \tag{3}$$

where $M_5(u)$ is a certain linear combination of the fifth derivatives of the function u; $i = 0, 1, 2, 3$; $\left(\sum\right)_i$ is the sum over the three points neighboring i; $\delta = 1$ for $i \in A_h^1$ and $\delta = -1$ for $i \in B_h^1$.

Let, on the set A_h^1, for example at the point 0,

$$\Delta f_0 = a_4 \Delta f_0 + a_5 \Delta_h' f_0 + a_6 \Delta_h'' f_0 + a_5 O(h) + O(h^2), \tag{4}$$

and on the set B_h^1, for example at the points $i = 1, 2, 3$,

$$\Delta f_i = b_4 \Delta f_i + b_5 \Delta_h' f_i + b_6 \Delta_h'' f_i + b_5 O(h) + O(h^2), \tag{5}$$

where

$$a_4 + a_5 + a_6 = 1, \tag{6}$$

$$b_4 + b_5 + b_6 = 1. \tag{7}$$

We shall seek difference schemes in the form

$$\Delta'_h u_0 = a_1 f_0 + a_2 \left(\sum f\right)_0 + a_3 h^2 \Delta f_0, \tag{8}$$

$$\Delta'_h u_i = b_1 f_i + b_2 \left(\sum f\right)_i + b_3 h^2 \Delta f_i, \tag{9}$$

$$i = 1, 2, 3.$$

To determine the coefficients $a_i, b_i,\ i = 1, 2, 3, \dots, 6$, note that

$$3\Delta'_h u_0 + \left(\sum \Delta'_h u\right)_0 = 6\Delta''_h u_0 = 3h^2(a_3 + b_3)\Delta f_0 + \frac{9}{2} b_2 h^2 \Delta''_h f_0 +$$

$$+\frac{3}{4}(3a_2 + b_1)h^2 \Delta'_h f_0 + 3f_0(a_1 + b_1 + 3(a_2 + b_2)) + O(h^4), \tag{10}$$

$$3\Delta'_h u_i + \left(\sum \Delta'_h u\right)_i = 6\Delta''_h u_i = 3h^2(a_3 + b_3)\Delta f_i + \frac{9}{2} a_2 h^2 \Delta''_h f_i +$$

$$+\frac{3}{4}(3b_2 + a_1)h^2 \Delta'_h f_i + 3f_i(a_1 + b_1 + 3(a_2 + b_2)) + O(h^4). \tag{11}$$

Using the equalities (1)–(5), (10), (11), we obtain for a_i, b_i the equalities

$$a_1 + b_1 + 3(a_2 + b_2) = 2, \qquad a_3 + b_3 = \frac{3}{8} a_4, \qquad 3a_2 + b_1 = \frac{3}{2} a_5, \qquad 4b_2 = a_6; \tag{12}$$

$$a_3 + b_3 = \frac{3}{8} b_4, \qquad 3b_2 + a_1 = \frac{3}{2} b_5, \qquad 4a_2 = b_6. \tag{13}$$

Equations (6), (7), (12), (13) will serve as the starting point for forming systems of equations.

1) Let us require that schemes (8), (9) be identical, i.e., that $a_i = b_i$, $i = 1, 2, \dots, 6$; then from the system of equations (6), (7), (12), (13) we obtain

$$a_1 = {}^3\!/_4(1 + a_4), \quad a_2 = {}^1\!/_4({}^1\!/_3 - a_4), \quad a_3 = {}^3\!/_{16} a_4, \quad a_5 = {}^2\!/_3, \quad a_6 = {}^1\!/_3 - a_4. \tag{14}$$

From the whole variety of difference schemes obtained, let us choose two: setting $a_4 = 0$, we obtain

$$\Delta'_h u_i = {}^3\!/_4 f_i + {}^1\!/_{12} \left(\sum f\right)_i, \tag{15}$$

and setting $a_4 = {}^1\!/_3$, we obtain

$$\Delta'_h u_i = f_i + \frac{h^2}{16}\Delta f_i, \quad i = 0, 1, 2, 3. \tag{16}$$

If at the points of the set A_h^2 the function u is given exactly and the functions u and f have bounded fifth and third derivatives, respectively, then schemes (14), (15), (16) give an error in the approximate solution of order $O(h^3)$; this follows from the fact that equation (2) is satisfied with the same accuracy.

2) Let us find difference schemes that give an error in the solution $O(h^4)$ on the set A_h^1 and $O(h^3)$ on the set B_h^1, under the condition that the functions u and f have bounded sixth and fourth derivatives, respectively. For this purpose consider the systems of equations (6), (7), (12), (13), in which we put $a_5 = 0$; then, expressing the unknowns in terms of a_2, a_3, we obtain

$$a_1 = 1 - 3a_2, \quad a_4 = b_4 = -({}^1\!/_3 + 4a_2), \quad a_6 = 4({}^1\!/_3 + a_2), \quad b_1 = -3a_2,$$

$$b_2 = {}^1\!/_3 + a_2, \quad b_3 = -{}^1\!/_8 - {}^3\!/_2 a_2 - a_3, \quad b_5 = {}^4\!/_3, \quad b_6 = 4a_2. \tag{17}$$

From the whole variety of schemes let us present one: for $a_2 = -{}^1\!/_{12}$, $a_3 = 0$,

$$\Delta'_h u_0 = {}^5\!/_4 f_0 - {}^1\!/_{12} \left(\sum f\right)_0, \quad \Delta'_h u_i = {}^1\!/_4 f_i + {}^1\!/_4 \left(\sum f\right)_i, \quad i = 1, 2, 3. \tag{18}$$

3. Finally, one may require that equation (11) be satisfied with accuracy $O(h^2)$, and equation (10) with accuracy $O(h^4)$; then it is sufficient to find the general solution of the system (6), (12) for $a_5 = 0$; one such solution will be, for example, the following:

$$\Delta'_h u_0 = 2f_0 + {}^3\!/_8 h^2 \Delta f_0, \quad \Delta'_h u_i = 0, \quad i = 1, 2, 3. \tag{19}$$

It is obvious that all schemes (14), (17), (19), for $f \equiv 0$ and sufficient smoothness of the function u, give on the set A_h^1 an error of order $O(h^4)$. Schemes (14)–(19) are of special interest when the function f depends on u: for example, scheme (16) is convenient to apply when $f = \lambda u$; then

$$\Delta'_h u_i = \left(\lambda + \frac{h^2\lambda^2}{16}\right) u_i, \quad i = 0, 1, 2, 3.$$

Four-point schemes for nonstationary equations are obtained analogously; for example, for the equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

using (15), one can give the scheme

$$3(r + {}^1\!/\!_4)u_i + ({}^1\!/\!_{12} - r)\left(\sum u\right)_i\Big|_{t+\Delta t} = 3({}^1\!/\!_4 - r)u_i + ({}^1\!/\!_{12} + r)\left(\sum u\right)_i\Big|_t .$$

where $r = \sqrt[3]{\frac{\Delta t}{h^2}}$, which gives an error of order $O(\Delta t^2 + h^3)$; for $h^2 = 8\Delta t$ it becomes explicit

$$u_i\big|_{t+\Delta t} = \frac{1}{2}u_i + \frac{1}{6}\left(\sum u\right)_i\Big|_t .$$

Apparently, the schemes presented are expedient to use in multigroup reactor calculations (the plane case). In n-dimensional space, the generalization of four-point schemes will be $(n+2)$-point schemes approximating the Laplace operator on a regular system of points that are the centers and vertices of regular n-dimensional simplexes. In this case

$$\Delta'_h u_i \equiv \frac{2n}{(n+1)h^2}\left(\left(\sum u\right)_i - (n+1)u_i\right) = \Delta u_i + \delta h M_3(u_i) + O(h^2),$$

$$\Delta''_h u_i = \frac{n}{(n+1)^2h^2}\left(\left(\sum{}' u\right)_i - n(n+1)u_i\right) = \Delta u_i + O(h^2),$$

and, for $n > 2$, these schemes give an error $O(h^2)$, which cannot be improved in order.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.