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Abstract

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MATHEMATICS

K. S. SIBIRSKII

UNIFORM APPROXIMATION OF POINTS OF DYNAMIC LIMIT SETS AND OF MOTIONS IN THEM

(Presented by Academician P. S. Aleksandrov on 14 IV 1962)

In the theory of dynamical systems ^(1,2), an important role is played by motions whose dynamic limit sets Ω_p and A_p are nonempty.

V. V. Nemytskii ⁽³⁾ posed the problem of studying the character of the return of a trajectory to a neighborhood of its limit set and its influence on the properties of motions in this set. He also obtained ^(3,4) substantial results in this area and, in particular, showed that in order that the set Ω_p of all ω -limit points of a motion $f(p, t)$, positively stable in the sense of Lagrange, be a minimal set, it is necessary and sufficient that $f(p, I^+)$ uniformly approximate Ω_p ^(1,3).

In the present note, without assuming minimality of the ω -limit set, we determine which of its subsets are uniformly approximated by the semitrajectory $f(p, I^+)$, and establish necessary and sufficient conditions for almost periodicity and periodicity of motions in Ω_p .

1°. Let a dynamical system $f(p, t)$ be given in an arbitrary metric space R ⁽¹⁾. Following V. V. Nemytskii, we shall say that the semitrajectory $f(p, I^+)$ ($f(p, I^-)$) uniformly approximates the set Q if for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that every arc of the semitrajectory $f(p, I^+)$ ($f(p, I^-)$) of temporal length T approximates the set Q to within ε , i.e.

$$Q \subseteq S(f(p; t_0, t_0 + T), \varepsilon)$$

$$(Q \subseteq S(f(p; -t_0 - T, -t_0), \varepsilon))$$

for every $t_0 \geq 0$.

We shall call a point q a ψ -(β -)limit point of the motion $f(p, t)$ if the semitrajectory $f(p, I^+)$ ($f(p, I^-)$) uniformly approximates the point q . The set of all ψ -(β -)limit points of the motion $f(p, t)$ will be called the ψ -(β -)limit set of the motion $f(p, t)$ and denoted by Ψ_p (B_p).

In what follows we shall speak only about the sets Ψ_p and Ω_p . All results can without difficulty be transferred to the sets B_p and A_p .

A sequence of nonnegative numbers $\{t_n\}$ is called relatively dense on $I^+ = [0, +\infty)$ if there exists an $L > 0$ such that, for every $l \geq 0$, the interval $[l, l + L]$ contains at least one point of this sequence.

It is clear that a point q is a ψ -limit point of the motion $f(p, t)$ if and only if, for every $\varepsilon > 0$, there exists a sequence $\{t_n\}$ relatively dense on I^+ such that

$$\bigcup_{n=1}^{\infty} f(p, t_n) \subseteq S(q, \varepsilon).$$

It follows directly from the definitions that

$$\Psi_p \subseteq \Omega_p \subseteq \overline{f(p, I)}.$$

It is also not difficult to show that the set Ψ_p is invariant and closed, and if p and q are two points of one trajectory, then $\Psi_p = \Psi_q$.

It is known that a motion $f(p, t)$ stable in the sense of Poisson in the positive direction is characterized by each of the following two relations:

$$f(p, I) \cap \Omega_p \neq \Lambda, \quad \Omega_p = \overline{f(p, I)}.$$

Similarly, the motion $f(p, t)$ is almost recurrent ⁽⁵⁾ if and only if

$$f(p, I) \cap \Psi_p \neq \Lambda$$

or, what ...

the same, $\Psi_p = \overline{f(p, I)}$. In particular, if the motion $f(p, t)$ is special (periodic or stationary), then $\Psi_p = \Omega_p = \overline{f(p, I)}$.

It is not difficult to show that if K is a nonempty compact subset of Ψ_p , then the positive semitrajectory $f(p, I^+)$ uniformly approximates K .

2°. We shall give two theorems characterizing the structure of the set Ψ_p .

Theorem 1. *The set Ψ_p is empty in each of the following two cases:*

1. Ω_p contains more than one minimal set.
2. The space R is locally compact, and the motion $f(p, t)$ is not Lagrange stable in the positive direction.

For the proof of this theorem it is enough to establish that the following two lemmas hold:

Lemma 1. *If Ω_p contains a minimal set M , then either $\Psi_p = M$, or $\Psi_p = \Lambda$.*

Lemma 2. *If an open set $U \subseteq R$ is such that \overline{U} is compact, and the motion $f(p, t)$ is not Lagrange stable in the positive direction, then for any $T > 0$ there exists $t_0 \geq 0$ such that $f(p; t_0, t_0 + T) \cap U = \Lambda$.*

Let us formulate two corollaries following from Theorem 1.

Corollary 1. *In a locally compact space Ψ_p is compact.*

Corollary 2. *In a locally compact space every almost recurrent motion is recurrent.*

A statement analogous to the latter for the case of a compact space is given in ⁽⁵⁾.

As is known, Lagrange stability in the positive direction of the motion $f(p, t)$ ensures the existence in Ω_p of at least one minimal set.

Theorem 2. *If the motion $f(p, t)$ is Lagrange stable in the positive direction, and M is the unique minimal set in Ω_p , then $\Psi_p = M$.*

Let us note that on the line $E^{(1)}$ one always has $\Psi_p = \Omega_p$. If, however, the dynamical system fills a region in the plane $E^{(2)}$ and is described by a system of two differential equations, then $\Psi_p \neq \Omega_p$ if and only if the semitrajectory $f(p, I^+)$ is bounded and Ω_p contains no more than one equilibrium point. Indeed, if Ω_p contained more than one equilibrium point, then, by Theorem 1, Ψ_p would be empty. If Ω_p contains one and only one equilibrium point q , then q is the only minimal set in Ω_p , since, besides the point q , Ω_p can contain at most a countable number of trajectories K_i approaching this point at both ends ^(1, 6). In this case $\Psi_p = q$. In the case when Ω_p contains no special points, it consists of one periodic trajectory $f(q, I)$ ^(1, 6), and then $\Psi_p = \overline{f(q, I)}$. From the two theorems given above there follows

Theorem 3. *In order that the positive semitrajectory $f(p, I^+)$ of a motion $f(p, t)$ Lagrange stable in the positive direction uniformly approximate some subset $Q \subseteq \Omega_p$, it is necessary and sufficient that $f(Q, I)$ be the unique minimal set in Ω_p .*

This theorem generalizes the result formulated in the introduction, due to V. V. Nemytskii. On the other hand, it establishes a new property of motions whose ω -limit set contains a unique minimal set. Such motions were introduced for consideration by G. D. Birkhoff under the name of positively Poisson-stable motions ⁽⁷⁾.

3°. We now formulate two theorems establishing conditions for almost periodicity and periodicity of motions in the ω -limit set. In connection with this we define the distance between two fundamental sequences $\{p_n\}$ and $\{q_n\}$ of points of the space R by the formula

$$\rho(\{p_n\}, \{q_n\}) = \lim_{n \rightarrow +\infty} \rho(p_n, q_n) \quad (8)$$

Theorem 4. *In order that, in the minimal ω -limit set of a motion $f(p, t)$ positively stable in the sense of Lagrange, all motions be almost periodic, it is necessary and sufficient that for every $\varepsilon > 0$ there exist a $\delta > 0$ such that from $\rho(\{f(p, t_n)\}, \{f(p, t'_n)\}) < \delta$ it follows that $\rho(\{f(p, t_n + t)\}, \{f(p, t'_n + t)\}) < \varepsilon$*

ε , whatever the number $t \geq 0$ and the fundamental sequences $\{f(p, t_n)\}$ and $\{f(p, t'_n)\}$ may be, such that $\{t_n\} \rightarrow +\infty$ and $\{t'_n\} \rightarrow +\infty$.

We note that the sufficiency of the condition stated in this theorem also follows without the assumption of minimality of Ω_p . In the case considered in (3), when the positive semitrajectory $f(p, I^+)$ is uniformly stable in the sense of Lyapunov in the positive direction relative to $f(p, I^+)$, the condition stated in Theorem 4 is, obviously, satisfied.

Theorem 5. In order that the ω -limit set of a motion $f(p, t)$ positively stable in the sense of Lagrange represent the trajectory of a special motion, it is necessary and sufficient that for each point $q \in \Omega_p$ there exist a sequence $\{t_n\}$ relatively dense on I^+ such that $\{f(p, t_n)\} \rightarrow q$.

In proving this theorem we use the fact that if $\{f(p, t_n)\} \rightarrow q$ and $\{t_n\}$ is relatively dense on I^+ , then the motion $f(q, t)$ is special. An analogous proposition for the particular case of a dynamical system was established by M. I. Almukhamedov (9).

Institute of Physics and Mathematics
Academy of Sciences of the MSSR

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