ON THE SUMMATION OF ORTHOGONAL SERIES BY EULER METHODS

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Abstract

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MATHEMATICS

O. A. ZIZA

ON THE SUMMATION OF ORTHOGONAL SERIES BY EULER METHODS

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Consider an orthogonal series from L^2, i.e., a series of the form

$$\sum_{n=0}^{\infty} a_n \varphi_n(x), \qquad x \in [a,b], \tag{1}$$

where $\{\varphi_n(x)\}$, $n = 0, 1, ...$, is an orthonormal (ON) system of functions on the interval $[a,b]$, and

$$\sum_{n=0}^{\infty} a_n^2 < \infty. \tag{2}$$

We introduce two definitions.

Definition 1*. Two summation methods T_1 and T_2 are called **equivalent in the space** L^2 if, from the summability of every orthogonal series from L^2 (1) by one of these methods on any set D of positive measure, there follows the summability of this series by the other method almost everywhere on D.

Definition 2**. Let $\{n_m\}$, $m = 0, 1, ...$, be an increasing sequence of natural numbers. We shall say that the numerical series

$$\sum_{n=0}^{\infty} u_n \tag{3}$$

with partial sums U_n is **summed by the method** $T[n_m]$ to the sum S, if $\lim_{m\to\infty} U_{n_m} = S$.

It is known that all Cesàro methods (C, α), $\alpha > 0$, are equivalent in the space L^2 to the method $T[2^m]$. For $\alpha = 1$ this theorem was proved by S. Kaczmarz; for any $\alpha > 0$, by A. Zygmund (^{2}), pp. 219 and 222).

An analogous theorem was also proved by A. Zygmund for methods of normal Riesz means (^{3}). Recall that the series (3) is summed to the sum S by the method of normal Riesz means (R, P_n, δ), where $0 < P_n < P_{n+1} \to \infty$, if

$$\lim_{\omega \to \infty} \sum_{P_n < \omega} \left(1 - \frac{P_n}{\omega}\right)^{\delta} u_n = S$$

(see (^{4}), p. 115).

A. Zygmund' s result is as follows: the Riesz methods (R, P_n, δ) for all $\delta > 0$ are equivalent in the space L^2 to the method $T[n_m]$, where the numbers n_m are determined from the condition

$$P_{n_m} \leqslant 2^m < P_{n_m+1}.$$

D. E. Menshov showed (^{1}) that for an arbitrary summation method a theorem of this type does not hold: there exists a matrix perfectly regular—

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* This definition differs somewhat from Definition 1 of paper (^{1}), p. 353.
** See (^{1}), Definition 2, p. 353.

method T, not equivalent in the space L^2, in the sense of Definition 1 given above, to any method of the form $T[n_m]$*.

In the present paper we consider the summation of orthogonal series from L^2 by the Euler and Borel methods.

The series (3) is summable to the sum S by the Euler method (E, q), $0 < q < \infty$, if

$$\lim_{n \to \infty} E_n^q = S,$$

where

$$E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} U_k.$$

The series (3) is summable to S by the exponential Borel method B, if

$$\lim_{x \to \infty} B(x) = S,$$

where

$$B(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} U_n,$$

moreover the latter series converges for all $x \geq 0$ ((4), pp. 224 and 229)**.

Let us formulate the main theorem.

Theorem 1. In the space L^2, all Euler methods (E, q), $q > 0$, and the Borel method are equivalent to one another and equivalent to the method $T[m^2]$.

We note that summability of orthogonal series by Euler methods was considered by Yu. Meder (5, 6). In particular, he obtained the following results: 1) from the summability almost everywhere on $[a, b]$ of the series (1) by any Euler method (E, q) there follows the summability almost everywhere of this series by the method $T[2^m]$; 2) there exists an orthogonal series from L^2, summable almost everywhere on $[a, b]$ by the method $T[2^m]$ and almost nowhere summable by the method (E, q) for any $q > 0$.

Let us outline the proof of Theorem 1. Denote the (E, q)-mean of the series (1) by $E_n^q(x)$.

Lemma 1. The series

$$\sum_{n=1}^{\infty} \sqrt{n}\left[E_n^q(x) - E_{n-1}^q(x)\right]^2$$

converges almost everywhere on $[a, b]$ for every $q > 0$.

From this lemma follows the equivalence in L^2 of all methods (E, q) and the method B. Indeed***, if the series (1) is summable on a set D, $mD > 0$, by the method B, then, by Szasz' s theorem ((7), § 4), the method B also sums on D the (E, q)-means of this series. From the latter fact and Lemma 1, using the Tauberian theorem of P. Schmidt (see, for example, (4), p. 385), we obtain convergence almost everywhere on D of the sequence $\{E_n^q(x)\}$, i.e. summability of the series (1) by the method (E, q). Since the method B is stronger than all methods (E, q) (see, for example, (4), p. 230), it follows that all methods (E, q) and the method B are equivalent in L^2.

Let us now outline the proof of the equivalence of the methods $(E, 1)$ and $T[m^2]$. For this we shall need the following modification of the method (R, P_n, δ). We shall say that the series (3) is summable to S by the method $(\bar{R}, p_n)$, if $p_0 > 0$,

$$p_n \geq 0, \qquad P_n = \sum_{k=0}^{n} p_k \to \infty$$

and

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=0}^{n} p_{kU}k = S$$

((4), p. 79).

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* This follows from the following. The method constructed in (1) is such that for any sequence $\{n_m\}$ either there is a series of the form (1), summable by this method almost everywhere but not summable almost nowhere by the method $T[n_m]$, or there is a series of the form (1), summable almost everywhere by the method $T[n_m]$ but not summable almost nowhere by the constructed method ((1), pp. 405 and 418).

** In what follows, only the **exponential** Borel method is considered.

*** This argument is analogous to the proof of equivalence in L^2 of the methods (C, α), $\alpha \geq 1$, and the Abel method ((2), p. 220).

The theorem of A. Zygmund cited above, as well as certain estimates for the $(R, \overline{P}_n, \delta)$-means of series (1) from paper (^{3}), are also valid for the corresponding methods $(\overline{R}, p_n)$. (In particular, for $p_n = \frac{1}{n}$, see also (^{8}).)

Lemma 2. There exists a Riesz method $(\overline{R}, q_n)$ which is weaker* than the method $(E, 1)$ and which is equivalent in the space L^2 to the method $T[m^2]$.

Lemma 3. There exists a Riesz method $(\overline{R}, t_n)$, equivalent in the space L^2 to the method $T[m^2]$, and such that if the method $(E, 1)$ sums series (3), then it also sums the $(\overline{R}, t_n)$-means of this series.

It follows at once from Lemma 2 that from the summability of series (1) on a set of positive measure by the method $T[m^2]$, there follows the summability almost everywhere on this set of series (1) by the method $(E, 1)$. The converse assertion is obtained from Lemma 3 by arguments analogous to those given above (after Lemma 1). In doing so one uses the estimates (somewhat modified) for the Riesz means of series (1) given in paper (^{3}).

We shall give one more theorem, which is a consequence of Theorem 1.

Definition 3. A function $\Omega(n) > 0$ is called an **exact Weyl multiplier for the summability method** T if: 1) from the condition $\sum c_n^2 \Omega(n) < \infty$ there follows the summability almost everywhere by the method T of the series $\sum c_n \varphi_n(x)$, where $\{\varphi_n(x)\}$ is an arbitrary ON-system on $[a, b]$; 2) for every function $0 < \omega(n) = o(\Omega(n))$ there exists an orthogonal series $\sum b_n \psi_n(x)$, not summable by the method T at any point of the interval $[a, b]$, and such that $\sum b_n^2 \omega(n) < \infty$.

As is known, an exact Weyl multiplier for convergence is the function $\ln^2 n$. These are the theorems of D. E. Menshov—H. Rademacher and D. E. Menshov (see, for example, (^{2}), pp. 190 and 195).

A. Zygmund in paper (^{3}) found exact Weyl multipliers for the methods (R, P_n, δ). (A correction to Zygmund' s formulation was indicated by Chen Jiang-gong ((^{9}), p. 14).)

Theorem 2. An exact Weyl multiplier for the Euler methods (E, q) and for the Borel method is the function $\ln^2 n$.

This theorem follows from Theorem 1 and the following lemma.

Lemma 4. If the method T is equivalent in the space L^2 to some method $T[n_m]$, then an exact Weyl multiplier for the method T is the function

$$\Omega(n) = \ln^2 m \quad \text{for } n_m \leq n < n_{m+1}, \qquad m = 1, 2, \dots .$$

(For $0 \leq n < n_1$ the values of $\Omega(n)$ may be defined arbitrarily.)

Lemma 4 follows easily from the theorems on exact Weyl multipliers formulated above.

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All-Union Correspondence
Power Engineering Institute

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* The case of equivalence is not excluded here.

Note: Figure translations are in progress. See original paper for figures.

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