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Abstract

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MATHEMATICS

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REPRESENTATION OF SEMIGROUPS BY MEANS OF BINARY RELATIONS

(Presented by Academician A. I. Mal'cev on 2 X 1961)

In the present note the construction of an arbitrary representation of a given abstract semigroup by means of binary relations is described. The problem of finding all representations of a given semigroup by binary relations of a special kind (transformations, partial transformations) was solved by V. V. Wagner ⁽¹⁾ and, in another way, by E. S. Lyapin ⁽²⁾.

We use the notation of mathematical logic: \wedge is the symbol of conjunction, \rightarrow the symbol of implication, \leftrightarrow the symbol of logical equivalence, \bigwedge the universal quantifier, \bigvee the existential quantifier.

A **binary relation between elements of a set** A is any subset ρ of the set $A \times A$. For binary relations between elements of the set A all set-theoretic operations are defined.

To binary relations $\rho_1 \subset A \times A$ and $\rho_2 \subset A \times A$ there corresponds their **product** — the binary relation $\rho_2 \circ \rho_1 \subset A \times A$, defined by the formula

$$(a_1, a_2) \in \rho_2 \circ \rho_1 \leftrightarrow \bigvee (a) [(a_1, a) \in \rho_1 \wedge (a, a_2) \in \rho_2]. \quad (1)$$

The set $\mathfrak{P}(A \times A)$ of all binary relations between elements of the set A is a semigroup with respect to the operation of multiplication of binary relations. The zero of this semigroup is the empty binary relation \emptyset , and the identity is the identity binary relation Δ_A .

A **section of a binary relation** $\rho \subset A \times A$ **at an element** $a_0 \in A$ is the subset $\rho \langle a_0 \rangle$ of the set A defined by the formula

$$a \in \rho \langle a_0 \rangle \leftrightarrow (a_0, a) \in \rho. \quad (2)$$

In the case when each section of a binary relation ρ by elements of the set A contains no more than one element, the binary relation ρ is called **single-valued**, or a **transformation of the set A** . If in this case for all $a \in A$, $\rho\langle a \rangle \neq \emptyset$, then ρ is called a **complete transformation of the set A** ^{*}. In the case when ρ is a transformation of the set A and $(a_1, a_2) \in \rho$, one writes: $\rho(a_1) = a_2$.

To every binary relation $\rho \subset A \times A$ one can put in correspondence a single-valued binary relation $\check{\rho} \subset \mathfrak{P}(A) \times \mathfrak{P}(A)$, **called the extension of the binary relation**^{**} ρ **to the set of subsets** and defined by the formula

$$\check{\rho}(a) = \bigcup_{a \in a} \rho\langle a \rangle \quad \text{for } a \subset A. \quad (3)$$

To every binary relation $\rho \subset A \times A$ we put in correspondence a binary relation $\hat{\rho} \subset \mathfrak{P}(A) \times \mathfrak{P}(A)$, called the **majorant extension of the binary relation ρ to the set of subsets**—

* Often transformations are called partial transformations, and complete transformations are called (simply) transformations.

** $\mathfrak{P}(A)$ is the set of subsets of the set A .

the set of subsets and defined by the formula

$$(a_1, a_2) \in \hat{\rho} \leftrightarrow \bar{\rho}(a_1) \subset a_2. \quad (4)$$

The majorant extension of binary relations defines a mapping of the semigroup $\mathfrak{P}(A \times A)$ into the semigroup $\mathfrak{P}(\mathfrak{P}(A) \times \mathfrak{P}(A))$. It turns out that this mapping is an isomorphism, i.e.

$$\hat{\rho}_2 \circ \hat{\rho}_1 = \widehat{\rho_2 \circ \rho_1}, \quad \hat{\rho}_1 = \hat{\rho}_2 \rightarrow \rho_1 = \rho_2. \quad (5)$$

The **restriction** of a binary relation $\rho \subset A \times A$ **to a subset** $a \subset A$ is a binary relation $\rho_a \subset a \times a$ such that $\rho_a = \rho \cap a \times a$.

A **representation of a semigroup G by means of binary relations between elements of a set A** is a homomorphism of the semigroup G into the semigroup $\mathfrak{P}(A \times A)$. In the case of an isomorphism, the representation is called **faithful**.

Thus, a mapping P of the semigroup G into $\mathfrak{P}(A \times A)$ will be a representation if and only if^{*}

$$P(g_2) \circ P(g_1) = P(g_1 g_2). \quad (6)$$

If the images of the elements of the semigroup G under the representation P are (full) transformations, then P is called a **representation by means of (full) transformations**.

The kernel of a representation P is the stable equivalence relation ε_P between elements of the semigroup G , defined as follows:

$$g_1 \equiv g_2(\varepsilon_P) \leftrightarrow P(g_1) = P(g_2). \quad (7)$$

The image of the semigroup G under the representation P is isomorphic to the quotient semigroup G/ε_P .

Adjoining to the semigroup G an element e not belonging to the set G , and defining $ee = e$, $eg = ge = g$ for all $g \in G$, we obtain a semigroup G^* with identity e .

The representation Λ of the semigroup G by means of full transformations of the set G^* , defined by the formulas

$$(g_1, g_2) \in \Lambda(g) \leftrightarrow g_2 = g_1g, \quad (e, g) \in \Lambda(g), \quad (8)$$

is called the **canonical representation** of the semigroup G **by means of right shifts**. As is known, the representation Λ is faithful.

Let θ be a one-to-one mapping of the set A onto the set B . To every binary relation $\rho \subset A \times A$ we associate the binary relation $\dot{\rho} \subset B \times B$, defined by the formula $(a_1, a_2) \in \rho \leftrightarrow (\theta(a_1), \theta(a_2)) \in \dot{\rho}$. If P is a representation of the semigroup G by means of binary relations between elements of the set A , then, defining $\dot{P}(g) = \overline{\dot{P}(g)}$, we obtain a representation \dot{P} of the semigroup G by means of binary relations between elements of the set B . The representations P and \dot{P} are called **similar**. It is clear that similar representations differ from one another inessentially, since we are not interested in the nature of the elements of the set A between whose elements the binary relations are taken.

Let $\{\Lambda_i\}_{i \in I}$ be a family of representations of the semigroup G , each of which is similar to the canonical representation Λ , and let Λ_i be a representation by means of full transformations of the set A_i and of the set

* Sometimes formula (6) is replaced by the condition $P(g_1) \circ P(g_2) = P(g_1g_2)$; however, we consider definition (6) more acceptable, since the factors in the product of binary relations are written from right to left.

from the family $\{A_i\}_{i \in I}$ are pairwise disjoint. Then Λ_I , defined by the formula

$$\Lambda_I(g) = \bigcup_{i \in I} \Lambda_i(g), \quad (9)$$

will be a representation of the semigroup G by complete transformations of the set $\bigcup_{i \in I} A_i$. This representation is called the **canonical w -fold representation** of the semigroup G (here w is the cardinal number of the set I). The semigroup G has various canonical w -fold representations, but all these representations are similar.

Let P be a representation of the semigroup G by binary relations between the elements of a set A . Define $\hat{P}(g) = \widehat{P(g)}$. Then, by formulas (5), \hat{P} will be a representation of the semigroup G by binary relations between the elements of the set $\mathfrak{P}(A)$, and $\varepsilon_{\hat{P}} = \varepsilon_P$. The representation \hat{P} will be called the **derivative** of the representation P .

Let \mathfrak{a} be a subset of the set A . Construct a mapping $P_{\mathfrak{a}}$ of the semigroup G into the semigroup $\mathfrak{P}(\mathfrak{a} \times \mathfrak{a})$, defining $P_{\mathfrak{a}}(g) = (P(g))_{\mathfrak{a}}$. If $P_{\mathfrak{a}}$ proves to be a representation of the semigroup G by binary relations between the elements of the set \mathfrak{a} , then $P_{\mathfrak{a}}$ is called a **restriction** of the representation P to the set \mathfrak{a} .

Theorem 1. *Every representation of a semigroup by binary relations is similar to a restriction of the derivative of a canonical w -fold representation of the given semigroup, for some w .*

We shall give the plan of the proof of this theorem, without going into technical details.

Denote by $H_{a_1}^a$, where $a, a_1 \in A$, the subset of the set G^* defined by the formulas

$$g \in H_{a_1}^a \leftrightarrow (a, a_1) \in P(g), \quad e \in H_{a_1}^a \leftrightarrow a = a_1. \quad (10)$$

Let θ_a be a one-to-one mapping of the set G^* onto some set B_a , and let the sets of the family $\{B_a\}_{a \in A}$ be pairwise disjoint. To each θ_a there corresponds a similarity of a representation Λ and of some representation Λ_a . To the family of representations $\{\Lambda_a\}_{a \in A}$ there corresponds a canonical w -fold representation Λ_A , where w is the cardinal number of the set A . Denote by B^{a_1} the subset of the set $B = \bigcup_{a \in A} B_a$, defined as follows: $B^{a_1} = \bigcup_{a \in A} \theta_a(H_{a_1}^a)$. Let \mathfrak{A} be the set of all B^a (for all possible $a \in A$). It turns out that

$$(B^{a_1}, B^{a_2}) \in \hat{\Lambda}_{A\mathfrak{A}}(g) \leftrightarrow (a_1, a_2) \in P(g) \quad (11)$$

and from $B^{a_1} = B^{a_2}$ it follows that $a_1 = a_2$. Therefore the representations P and $\hat{\Lambda}_{A\mathfrak{A}}$ are similar.

The least of the cardinal numbers w such that P is similar to a restriction of the derivative of a canonical w -fold representation is called the **weight** of the representation P .

A representation P of a semigroup G by binary relations is called a **universal representation of weight w** if the weight of P is equal to w and every representation of the semigroup G of weight not exceeding w is similar to a restriction

of the representation P . It follows from Theorem 1 that the representation $\widehat{\Lambda}_I$, where the cardinal number of the set I is equal to w , will be a universal representation of the semigroup of weight w . It would be interesting to determine which semigroups can possess several non-similar universal representations of one and the same weight.

Let $\mathfrak{H} = \{H_a^i\}_{a \in A}^{i \in I}$ be a family of subsets of the semigroup G^* , specified by means of two index sets I and A . Of course, one and the same subset (in particular, the empty one) may occur in the family \mathfrak{H} several times.

each time with different indices. The family \mathfrak{H} is called **right admissible** for the semigroup G if, for any $g_1, g_2 \in G$, $a_1, a_2 \in A$,

$$(\bigwedge i)[H_{a_1}^i\{g_1g_2\} \subset H_{a_2}^i] \implies (\bigvee a)(\bigwedge i)[H_{a_1}^i\{g_1\} \subset H_a^i \wedge H_a^i\{g_2\} \subset H_{a_2}^i]. \quad (12)$$

Define a mapping $P_{\mathfrak{H}}$ of the semigroup G into the semigroup $\mathfrak{P}(A \times A)$:

$$(a_1, a_2) \in P_{\mathfrak{H}}(g) \leftrightarrow (\bigwedge i)[H_{a_1}^i\{g\} \subset H_{a_2}^i]. \quad (13)$$

Theorem 2. The mapping $P_{\mathfrak{H}}$ will be a representation of the semigroup G by means of binary relations if and only if the family \mathfrak{H} is right admissible for the semigroup G .

For the proof it suffices to write formula (6), substituting into it the expressions from (13).

Theorem 3. For every representation P of the semigroup G by means of binary relations there exists a family \mathfrak{H} , right admissible for the semigroup G , such that $P = P_{\mathfrak{H}}$.

Proof. Let P be a representation of the semigroup G by means of binary relations between the elements of a set A . Consider the family $\mathfrak{H} = \{H_{a_2}^{a_1}\}_{a_1 \in A, a_2 \in A}$ of subsets of the semigroup G^* , whose elements are defined by formulas (10). Let $(a_1, a_2) \in P_{\mathfrak{H}}(g)$. This means that $(\bigwedge a)[H_{a_1}^a\{g\} \subset H_{a_2}^a]$. In particular, $H_{a_1}^{a_1}\{g\} \subset H_{a_2}^{a_1}$ and, since $e \in H_{a_1}^{a_1}$, $g \in H_{a_2}^{a_1}$. By formulas (10), $(a_1, a_2) \in P(g)$. Conversely, let $(a_1, a_2) \in P(g)$. If $g_1 \in H_{a_1}^a$, then $(a, a_1) \in P(g_1)$ and $(a, a_2) \in P(g) \circ P(g_1) = P(g_1g)$, or $g_1g \in H_{a_2}^a$, whence $H_{a_1}^a\{g\} \subset H_{a_2}^a$, i.e. $(a_1, a_2) \in P_{\mathfrak{H}}(g)$. Thus, $P = P_{\mathfrak{H}}$. Therefore the family \mathfrak{H} is right admissible for the semigroup G . It turns out that the **weight of the representation** P is equal to the least of the cardinal numbers of the upper index sets of families \mathfrak{H} such that $P = P_{\mathfrak{H}}$.

It would be interesting to find all right admissible families for a given semigroup.

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CITED LITERATURE

¹ V. V. Wagner, *Matem. sborn.*, **38**(80), No. 2, 203 (1956).

² E. S. Lyapin, *Matem. sborn.*, **52**(94), No. 1, 589 (1960).

Note: Figure translations are in progress. See original paper for figures.

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