Mathematics

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Abstract

Full Text

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ON THE STABILIZATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR A PARABOLIC EQUATION

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Consider the parabolic equation:

$$\frac{\partial u}{\partial t} = \sum_{k,l=1}^{n} a_{kl}(t) \frac{\partial^2 u}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k(t) \frac{\partial u}{\partial x_k} + g(t)u; \qquad (1)$$

$$a_{ij}(t) = a_{ji}(t), \qquad \sum_{k,l=1}^{n} a_{kl}(t)\alpha_k\alpha_l \geq \gamma(t) \sum_{k=1}^{n} \alpha_k^2, \qquad \gamma(t) > 0. \qquad (1')$$

The functions $a_{kl}(t)$, $b_k(t)$, $g(t)$ for all $k, l = 1, 2, \dots, n$ are assumed to be integrable in every finite interval of the argument t.

Introduce the notation:

$$A_{kl}(t) = \int_0^t a_{kl}(\tau)\, d\tau, \qquad B_k(t) = \int_0^t b_k(\tau)\, d\tau, \qquad G(t) = \int_0^t g(\tau)\, d\tau.$$

The symmetric matrix $A = \|A_{ij}(t)\|$ is uniformly positive definite for $t \geq t^* > 0$, since the parabolicity condition $(1')$, integrated from 0 to t, gives

$$\sum_{k,l=1}^{n} A_{kl}(t)\alpha_k\alpha_l \geq \int_0^t \gamma(\tau)\, d\tau \sum_{k=1}^{n} \alpha_k^2.$$

The eigenvalues of A satisfy

$$\lambda_i(t) \geq \int_0^t \gamma(\tau)\, d\tau \geq \delta > 0 \quad \text{for} \quad t \geq t^* > 0.$$

The fundamental solution of equation (1), according to ([1]), can be written in the form

$$W(\bar{x}-\bar{\xi},t)=\frac{e^{G(t)}}{(2\pi)^n}\int \exp\left[i(\bar{y}',\bar{\alpha})-\bar{\alpha}'A\bar{\alpha}\right]\,d\alpha, \tag{2}$$

where the integral is extended over the whole space, $d\alpha = d\alpha_1 \dots d\alpha_n$,

$$\bar{\alpha}=\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix},\qquad \bar{\alpha}'=(\alpha_1,\dots,\alpha_n),\qquad \sum_{k=1}^{n}\alpha_k^2=(\bar{\alpha}',\bar{\alpha}),$$

$$\bar{y}=\begin{pmatrix}B_1(t)+x_1-\xi_1\\ \cdots \\ B_n(t)+x_n-\xi_n\end{pmatrix}\equiv \bar{B}+\bar{x}-\bar{\xi}.$$

With the aid of (2), the solution of the Cauchy problem for equation (1) with bounded initial condition

$$u\big|_{t=0}=\varphi(x_1,\dots,x_n)\equiv\varphi(\bar{x}) \tag{3}$$

can be written as follows:

$$u(t,\bar{x})= \tag{4}$$

$$=\frac{e^{G(t)}}{2^n\sqrt{\pi^n\det A}}\int\varphi(\bar{\xi})\exp\left[-\frac{1}{4}\sum_{i,j=1}^{n}\bar{A}_{ij}(t)(B_i+x_i-\xi_i)(B_j+x_j-\xi_j)\right],$$

where $\bar{A}_{ij}(t)$ are the elements of the matrix A^{-1}. For what follows, it is more convenient to present (4) in the form

$$u(t,\bar{x})=\frac{e^{G(t)}}{\sqrt{\pi^n}}\int\varphi(\bar{B}+\bar{x}-2PA\bar{\xi})\exp[-(\bar{\xi}',\bar{\xi})]\,d\bar{\xi}, \tag{*}$$

where

$$\Lambda^1\equiv\begin{pmatrix}\lambda_1(t) & & 0\\ & \ddots & \\ 0 & & \lambda_n(t)\end{pmatrix}=P'AP$$

is the Jordan normal form of the symmetric positive-definite matrix A; $P = \|p_{ij}(t)\|$ is an orthogonal matrix. Denote by $\lambda_{\max}(t)$ and $\lambda_{\min}(t)$, respectively, the maximum and minimum eigenvalues of the matrix A; then the estimate holds:

$$\exp\left[-\frac{1}{4\lambda_{\min}(t)}\right]\sum_{i=1}^{n}(B_i+x_i-\xi_i)^2 \leq$$

$$\leq \exp\left[-\frac{1}{4}\sum_{i,j=1}^{n}\bar{A}_{ij}(t)(B_i+x_i-\xi_i)(B_j+x_j-\xi_j)\right] \leq$$

$$\leq \exp\left[-\frac{1}{4\lambda_{\max}(t)}\sum_{i=1}^{n}(B_i+x_i-\xi_i)^2\right]. \tag{5}$$

Let $\bar{e}$ be the vector $(\varepsilon_1, \dots, \varepsilon_n)$, where each ε_i, for all $i = 1, 2, \dots, n$, can take only two different values ± 1; $R_{\bar{e}}$ is the angle of the space $\{\varepsilon_1 x_1 \geq 0, \dots, \varepsilon_{nx} n \geq 0\}$; $\bar{a}$ is an arbitrary vector $(a_1, \dots, a_n)$, with

$$[a] = \prod_{i=1}^{n} a_i;$$

$R_{\bar{e}a}$ is the parallelepiped $\{0 \leq \varepsilon_1 x_1 \leq a_1, \dots, 0 \leq \varepsilon_{nx} n \leq a_n\}$.

Following (2), we shall say that $\varphi(\bar{x})$ has angular limiting means l, if

$$E(\varphi) \equiv \lim_{\substack{a_1 \to +\infty \\ \dots \\ a_n \to +\infty}} \frac{1}{[a]} \int_{R_{\bar{e}a}} \varphi(\bar{\xi})\, d\bar{\xi} = l \tag{6}$$

for all vectors $\bar{e}$. We say that $\varphi(\bar{x})$ has a ball limiting mean equal to l, if

$$M(\varphi) \equiv \lim_{r \to +\infty} \frac{1}{c_n r^n} \int_0^r \int_\Omega \varphi(r'\bar{\omega})(r')^{n-1}\, d\Omega\, dr' = l, \tag{7}$$

where c_n is the volume of the n-dimensional unit ball; Ω is the n-dimensional unit sphere; $\bar{\omega}$ is a variable unit vector. We say that $\varphi(\bar{x})$ has a spherical limiting mean equal to l, if

$$N(\varphi) \equiv \lim_{r \to +\infty} \frac{1}{s_n r^{n-1}} \int_\Omega \varphi(r\bar{\omega}) r^{n-1}\, d\Omega = l, \tag{8}$$

where s_n is the area of the n-dimensional unit sphere Ω.

It can be proved that the existence, for a bounded function, of angular limiting means equal to l implies the existence for it of a spherical limiting mean equal to l. But, as the simple example

$$\varphi(x,y)=\begin{cases}1, & \text{for } x>0,\ y>0 \text{ and } x<0,\ y<0,\\ -1, & \text{for } x<0,\ y>0 \text{ and } x>0,\ y<0,\end{cases}$$

shows, the converse is not true. It is easily proved that if a bounded function has a spherical limiting mean equal to l, then it also has a ball limiting mean equal to l.

Let $u(t,\vec{x})$ be the solution of the Cauchy problem for equation (1) with initial condition (3). Suppose that $M(\varphi)=l$. The question arises: what conditions must be imposed on the coefficients of equation (1) so that the solution $u(t,\vec{x})$ stabilizes as $t\to+\infty$?

Theorem 1. If:

1) the trace of the matrix A, $\operatorname{Sp} A\to+\infty$ as $t\to+\infty$;
2) there exists K such that $(\operatorname{Sp} A)^n/\det A\le K$;
3)
$$\sum_{i=1}^{n} B_i^2(t)/\operatorname{Sp} A\to 0 \quad \text{as } t\to+\infty;$$
4)
$$\lim_{t\to+\infty} G(t)=c;$$
5) $\varphi(\vec{x})$ is bounded, $M(\varphi)=l$, and $\varphi(\vec{x})-l$ preserves its sign,

then the solution $u(t,\vec{x})$ of the Cauchy problem (1), (3) stabilizes to le^c, i.e.

$$\lim_{t\to+\infty} u(t,\vec{x})=le^c$$

uniformly in x in any bounded domain.

Theorem 2. If:

1) there exist a positive function $\varkappa(t)$ and a constant K such that
$$\varkappa(t)\sum_{i=1}^{n}\alpha_i^2\ge\sum_{k,l=1}^{n}a_{kl}(t)\alpha_k\alpha_l; \qquad \int_0^t\varkappa(\tau)\,d\tau-\int_0^t\gamma(\tau)\,d\tau\le K;$$
2)
$$\int_0^t\gamma(\tau)\,d\tau\to+\infty \quad \text{as } t\to+\infty;$$
3)
$$\sum_{i=1}^{n}B_i^2(t)\Big/\int_0^t\gamma(\tau)\,d\tau\to 0 \quad \text{as } t\to+\infty;$$

4)

$$\lim_{t\to+\infty} G(t) = c;$$

5) $\varphi(\vec{x})$ is bounded, $M(\varphi) = l$, then the solution of the Cauchy problem (1), (3) stabilizes to le^c uniformly in x in any bounded domain.

We give a brief proof of Theorem 2.

Let $\varphi(\vec{x}) - l = \psi(\vec{x})$. Clearly, $M(\psi) = 0$. Using (*), we have:

$$u(t,\vec{x}) - le^{G(t)} = \frac{e^{G(t)}}{\sqrt{\pi^n}} \int_\Xi \psi\Big(\vec{B} + \vec{x} - 2P\Lambda\vec{\xi}\Big) \exp\Big[-(\vec{\xi},\vec{\xi})\Big]\, d\vec{\xi} = I_1 + I_2, \tag{9}$$

where I_1 is the integral over the domain $(\delta > |\vec{\xi}|) \cup (|\vec{\xi}| > R)$, and I_2 over the remaining part of space $\delta < |\vec{\xi}| < R$. Since $\psi(\vec{x})$ is bounded, $|\psi(\vec{x})| \le M$, for every $\varepsilon > 0$ there exist δ^* and R^* such that $|I_1| < \varepsilon/3$ for $\delta < \delta^*$ and $R > R^*$. Passing in I_2 to spherical coordinates, separating out the ball mean and integrating by parts, we obtain:

$$I_2 = \frac{e^{G(t)}}{\sqrt{\pi^n}} \left\{ e^{-r^2} \int_0^r \int_\Omega \psi\Big(\vec{B} + \vec{x} - 2P\Lambda r'\vec{\omega}\Big)\,(r')^{n-1}\, d\Omega\, dr' \Big|_{r=\delta}^{r=R} \right.$$

$$\left. + \int_\delta^R 2re^{-r^2} \int_0^r \int_\Omega \psi\Big(\vec{B} + \vec{x} - 2P\Lambda r'\vec{\omega}\Big)\,(r')^{n-1}\, d\Omega\, dr'\, dr \right\} = \frac{e^{G(t)}}{\sqrt{\pi^n}} \{I_2' + I_2''\}. \tag{10}$$

For sufficiently small δ^* and $1/R^*$, $|I_2'| < \varepsilon/3$. Fixing $\delta < \delta^*$ and $R > R^*$, we make the change of variables $2r'\sqrt{\lambda_{\min}(t)} = \xi$ in the inner integral of the expression I_2'':

$$I_2'' = \int_\delta^R 2r^{n+1}e^{-r^2} \left\{ \frac{1}{\Big(2r\sqrt{\lambda_{\min}(t)}\Big)^n} \int_0^{2r\sqrt{\lambda_{\min}(t)}} \int_\Omega \psi(\bar{B} + \bar{x} - \xi P\tilde{\Lambda}\bar{\omega})\xi^{n-1}\, d\Omega\, d\xi \right\} dr,$$

where

$$\tilde{\Lambda} = \frac{1}{\sqrt{\lambda_{\min}(t)}}\Lambda.$$

The expression in braces is the integral of the function $\psi(\bar{x})$ over an n-dimensional ellipsoid with semiaxes $\{2r\sqrt{\lambda_1(t)}, \dots, 2r\sqrt{\lambda_n(t)}\}$ and center at the point $(\bar{B}(t)+\bar{x})$, rotated in some way in the space by the orthogonal matrix P. Using the conditions of the theorem and the fact that $M(\psi)=0$, one can show that the expression in braces tends to zero as $t\to+\infty$. Consequently, $|I_2|<2\varepsilon/3$ for sufficiently large t. The proof of the theorem now follows easily from this.

In particular, the conditions of Theorem 2 are satisfied for the equation

$$\frac{\partial u}{\partial t}=a(t)\Delta u+\sum_{k=1}^{n}b_k(t)\frac{\partial u}{\partial x_k},\qquad \text{where } a(t)>0,$$

$$\int_0^t a(\tau)\,d\tau\to\infty,\qquad \int_0^t b_k(\tau)\,d\tau\bigg/\left[\int_0^t a(\tau)\,d\tau\right]^{1/2}\to 0\quad \text{as } t\to\infty \text{ for } k=1,2,\dots,n.$$

Moreover, the following stabilization theorems hold:

Theorem 3. If:

1)
$$\lim_{t\to+\infty}G(t)=c;$$

2)
$$\lim_{|\bar{x}|\to+\infty}\varphi(\bar{x})=l;$$

3) there exists at least one k, $k=1,2,\dots,n$, such that

$$\lim_{t\to+\infty}\left|\frac{B_k(t)}{\sqrt{\operatorname{Sp}A}}\right|=+\infty,$$

then the solution of the Cauchy problem (1), (3) stabilizes to le^c as $t\to+\infty$, uniformly in x in any bounded domain.

Theorem 4. If:

1)
$$\lim_{t\to+\infty}G(t)=c;$$

2) $\gamma(t)$ in the parabolicity condition $(1')$ is such that

$$\int_0^t \gamma(\tau)\,d\tau\to+\infty\quad \text{as } t\to+\infty;$$

3) there exist constants K and l such that, for bounded $\varphi(\bar{x})$,

$$\left|\int_0^{x_1}\cdots\int_0^{x_n}[\varphi(\bar{\xi})-l]\,d\xi\right|\le K\qquad\text{for all }\bar{x},$$

then the solution of the Cauchy problem (1), (3) stabilizes to le^c as $t\to+\infty$, uniformly in x in any bounded domain.

Let us note that, in the case of one variable ($n=1$), condition 3) of Theorem 4 will be satisfied for every uniformly almost periodic function whose Fourier exponents do not have zero as an accumulation point (see ([3]), p. 89). In particular, this condition will be satisfied for a periodic function.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.