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Abstract

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ON THE HEATING AND MELTING OF A SOLID BODY BY FRICTION

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In the paper of the same title by S. S. Grigoryan ([1]), the problem of the heating and melting of a solid body streamed around by a high-velocity flow of a viscous liquid is considered in a formulation admitting a self-similar solution. Here the same problem is considered in a formulation that does not admit a self-similar solution. Using the method developed by us earlier ([2,3]), we reduce the problem to a system of nonlinear functional equations of Volterra type, solvable for small values by iteration.

§ 1. Let a solid body filling the strip $0 < x^* < l^*$ border, along the plane $x^* = l^*$, on a viscous incompressible liquid filling the half-space $x^* > l^*$. We shall assume that the solid body moves with velocity $v_0 = \text{const}$ parallel to the plane $x^* = l^*$, and that, owing to viscous dissipation of energy, the temperature at the boundary $x^* = l^*$ reaches, at the instant $t_0^* < 0$, the melting temperature of the solid body. In the subsequent process three phases will participate: the solid–filling the region $0 < x^* < y^*(t^*)$, the melt–the region $y^*(t^*) < x^* < l^*$, and the surrounding liquid–the region $l^* < x^* < \infty$. We assume that the state of the system is known for $t^* \leq 0$, with $y^*(0) < (0, l^*)$.

We shall assign the index $i = 1$ to the solid phase, $i = 2$ to the melt, and $i = 3$ to the surrounding liquid. Let $\vartheta_i, w_i, k_i^*, c_{pi}, \rho_i, a_i^{2*}$, and ν_i denote, respectively, the temperature, velocity, coefficient of thermal conductivity, specific heat, density, coefficient of thermal diffusivity, and viscosity of the i-phase. Let, further, λ^* be the latent heat of fusion per unit mass. We shall take the melting temperature to be zero. We assume $k_i^*, c_{pi}, \rho_i, a_i^{2*}, \nu_i$, and λ^* to be constant, $\rho_1 = \rho_2 = \rho$. Let $\psi_i^*(x^*)$ be the initial temperature of the i-phase, $f^*(t^*)$ the temperature at the boundary $x^* = 0$, and $\varphi_i^*(x^*)$ the initial velocity in the i-phase. Put

$$T = \max\{|\psi_i^*(x^*)|;\ |f^*(t^*)|;\ i = 1, 2, 3\} \tag{1.1}$$

and introduce dimensionless quantities

$$v_i = \frac{w_i}{v_0}; \qquad u_i = \frac{\vartheta_i}{T}; \qquad f = \frac{f^*}{T}; \qquad \psi_i = \frac{\psi_i^*}{T}; \qquad \varphi_i = \frac{\varphi_i^*}{v_0}; \qquad x = \frac{x^*}{l^*};$$

$$t = \frac{a_1^{2*} t^*}{l^{*2}}; \qquad y = \frac{y^*}{l^*}; \qquad l = \frac{y^*(0)}{l^*}; \qquad a_i^2 = \frac{a_i^{2*}}{a_1^{2*}}; \qquad b_i^2 = \frac{\nu_i}{a_1^{2*}}; \tag{1.2}$$

$$\gamma_i = \frac{\nu_i}{c_{pi} a_1^{2*}} \cdot \frac{v_0}{T}; \qquad \lambda_i = \rho_i \nu_i; \qquad k_i = \frac{k_i^* T}{\lambda \rho a_1^{2*}}.$$

In what follows, the index referring to the liquid phases will be omitted. The unknowns u_1, u, v, and y are to be determined from the conditions

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t} \quad \text{for } 0 < x < y(t); \qquad u_1\Big|_{x=0} = f(t); \qquad u_1\Big|_{x=y(t)} = 0;$$

$$u_1\Big|_{t=0} = \psi_1(x); \tag{1.3$_1$}$$

$$b^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad \text{for } y(t) < x < \infty; \qquad x \neq 1; \qquad v\Big|_{t=0} = \varphi(x); \qquad v\Big|_{x=y(t)} = 0;$$

$$v \text{ and } \lambda \frac{\partial v}{\partial x} \text{ are continuous for } x = 1; \tag{1.3$_2$}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - \gamma^2 \left(\frac{\partial v}{\partial x} \right)^2 \quad \text{for } y(t) < x < \infty; \quad x \neq 1;$$

$$u\Big|_{x=y(t)} = 0; \qquad u\Big|_{t=0} = \psi(x); \tag{1,3$_3$}$$

$$u \text{ and } k \frac{\partial u}{\partial x} \text{ are continuous for } x = 1$$

$$\frac{dy}{dt} = k_1 \frac{\partial u_1}{\partial x}\Big|_{x=y(t)-0} - k_2 \frac{\partial u_2}{\partial x}\Big|_{x=y(t)+0}; \qquad y(0) = l \subset (0,1). \tag{1,3$_4$}$$

We assume that

$$f(t) \leq -\delta < 0; \qquad \psi_1(x) \leq 0, \quad \psi(x) \geq 0 \quad \text{near } x = l. \tag{1,4}$$

§ 2. Let $w^*(x,t)$ be a solution, regular at infinity, of the equation

$$\alpha^2(x)\frac{\partial^2 w^*}{\partial x^2} = \frac{\partial w^*}{\partial t} - F(x,t); \qquad x \neq 1, \tag{2,1}$$

satisfying the conditions

$$w^* \text{ and } \mu(x)\frac{\partial w^*}{\partial x} \text{ are continuous for } x = 1. \tag{2,2}$$

Here

$$\alpha^2(x) = \begin{cases} \alpha_1^2 = \text{const}, & x < 1; \\ \alpha_2^2 = \text{const}, & x > 1; \end{cases} \qquad \alpha(x) = \begin{cases} \mu_1 = \text{const}, & x < 1; \\ \mu_2 = \text{const}, & x > 1. \end{cases} \tag{2,3}$$

By the fundamental solution $g(x, \xi, t-\tau \mid \alpha^2; \mu)$, corresponding to problem (2,1), (2,2), we shall mean the solution of the adjoint equation

$$\alpha^2(\xi)\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial g}{\partial \tau} = -\alpha^2(\xi)\delta(t-\tau), \qquad \tau < t, \tag{2, 4_1}$$

regular at infinity and satisfying the conjugation conditions

$$\alpha^2(\xi)\frac{\partial g}{\partial \xi} \text{ and } \mu^*(\xi)\alpha^2(\xi)g \text{ are continuous for } \xi = 1, \quad x \neq 1. \tag{2, 4_2}$$

Here $\mu^* = \mu_2$ for $\xi < 1$ and $\mu = \mu_1$ for $\xi > 1$; $\delta(x)$ is the Dirac δ-function. We must have

$$\begin{aligned} w^*(x,t) = & \int_{x_1(0)}^{x_2(0)} w^*(\xi, 0)\, g\left(x, \xi, t \mid \alpha^2; \mu\right)\, d\xi \\ & + \int_0^t \alpha^2(\xi)\frac{\partial}{\partial \xi} w^*(\xi,\tau)\, g\left(x,\xi,t-\tau \mid \alpha^2;\mu\right)\Big|_{\xi=-x_1(\tau)}^{x_2(\tau)} d\tau \\ & - \int_0^t w^*(\xi,\tau)\left[\alpha^2(\xi)\frac{\partial}{\partial \xi} g\left(x,\xi,t-\tau \mid \alpha^2;\mu\right) - \frac{d\xi}{d\tau} g\left(x,\xi,t-\tau \mid \alpha^2;\mu\right)\right]_{\xi=x_1(\tau)}^{x_2(\tau)} d\tau \\ & + \int_0^t d\tau \int_{x_1(\tau)}^{x_2(\tau)} F(\xi,\tau)\, g\left(x,\xi,t-\tau \mid \alpha^2;\mu\right)\, d\xi, \end{aligned} \tag{2,5}$$

if $\chi_i(t)$ are differentiable for $t > 0$, $\chi_1(t) < x < \chi_2(t)$, and $1 \subset (\chi_1, \chi_2)$.

Put

$$g_{11}\left(x,\xi,t\mid\alpha^2;\mu\right)=(2\alpha_1\sqrt{\pi t})^{-1}\exp\left(-\frac{(x-\xi)^2}{4\alpha_1^2 t}\right),$$

$$g_{12}\left(x,\xi,t\mid\alpha^2;\mu\right)=\frac{\mu_1\alpha_2-\mu_2\alpha_1}{\mu_1\alpha_2+\mu_2\alpha_1}\,g_{11}\left(2-x,\xi,t\mid\alpha^2;\mu\right), \tag{2,6}$$

$$g_{13}\left(x,\xi,t\mid\alpha^2;\mu\right)=\frac{2\alpha_1\mu_2}{\mu_1\alpha_2+\mu_2\alpha_2}\,\frac{1}{2\alpha_2\sqrt{\pi t}}\exp\left\{-\left(\frac{x-1}{\alpha_1}-\frac{\xi-1}{\alpha_2}\right)^2\frac{1}{4t}\right\}.$$

Next, define g_{14}, g_{15}, and g_{16} by the expressions g_{13}, g_{12}, and, respectively, g_{11}, replacing in them α_1 and μ_1 by α_2 and μ_2, and conversely. Finally, set:

$$g_i(x,\xi,t\mid\alpha^2;\mu)=\begin{cases} g_{11}-(-1)^i g_{12}, & \text{for } -\infty<x;\ \xi<1,\\ \zeta_i g_{13}, & \text{for } -\infty<x<1;\ 1<\xi<\infty,\\ \zeta_i^{-1} g_{14}, & \text{for } -\infty<\xi<1;\ 1<x<\infty,\\ (-1)^i g_{15}+g_{16}, & \text{for } 1<x;\ \xi<\infty.\end{cases}$$

Here

$$\zeta_1=1;\qquad \zeta_2=\alpha_1/\alpha_2. \tag{2,7}$$

By direct verification it is easy to see that $g_1(x,\xi,t-\tau\mid\alpha^2;\mu)$ is the fundamental solution corresponding to the problem (2,1), (2,2)*. We note that g_1 and g_2 are related by

$$\frac{\partial g_1}{\partial x}=-\frac{\partial g_2}{\partial \xi};\qquad \alpha^2(\xi)\frac{\partial^2 g_1}{\partial x\,\partial\xi}=\frac{\partial g_2}{\partial t}. \tag{2,8}$$

Let, finally, $G_i(x,\xi,t)$ be the Green' s functions of the first and second boundary-value problems for the heat-conduction equation on the half-line, i.e.

$$G_i(x,\xi,t)=\left\{\exp\left[-\frac{(x-\xi)^2}{4t}\right]+(-1)^i\exp\left[-\frac{(x+\xi)^2}{4t}\right]\right\}(2\sqrt{\pi t})^{-1}. \tag{2,9}$$

It is obvious that for a one-layer problem (i.e. for the case $\alpha_1=\alpha_2$), g_i in (2,7) may be replaced by G_i. In addition, G_i satisfy (2,10).

§ 3. Put

$$w(t) = \frac{\partial v}{\partial x}\Big|_{x=y(t)+0}; \qquad z_1(t) = \frac{\partial v_1}{\partial x}\Big|_{x=y(t)-0}; \qquad z_2(t) = \frac{\partial u}{\partial x}\Big|_{x=y(t)+0}. \quad (3,1)$$

We reduce problem (1,3) to a system of Volterra-type functional equations with respect to w, z, z_1, and y. Below we shall write

$$\frac{\partial v}{\partial x} = v_1(x,t);$$

$$g_i(x,\xi,t \mid a^2; k) = g_i(x,\xi,t); \qquad g_i(x,\xi,t \mid b^2; \lambda) = g_i^*(x,\xi,t). \qquad (3,2)$$

Using (2,7), the boundary conditions (1,3), and the notation (2,11), (3,1), and (3,2), we find that the following equalities must hold:

$$u_1 = \int_0^t f(\tau)\frac{\partial}{\partial \xi}G_1(x,0,t-\tau)\,d\tau + \int_0^l \psi_1(\xi)G_1(x,\xi,t)\,d\xi +$$

$$+ \int_0^t z_1(\tau)G_1(x,y(\tau),t-\tau)\,d\tau; \qquad (3,3_1)$$

$$v = -b_2^2\int_0^t w(\tau)g_1^*(x,y(\tau),t-\tau)\,d\tau + \int_l^\infty \varphi(\xi)g_1^*(x,\xi,t)\,d\xi; \qquad (3,3_2)$$

$$u = -a_2^2\int_0^t z(\tau)g_1(x,y(\tau),t-\tau)\,d\tau + \int_l^\infty \psi(\xi)g_1(x,\xi,t)\,d\xi +$$

$$+ \int_0^t d\tau \int_{y(\tau)}^\infty \gamma^2(\xi)v_1^2(\xi,\tau)g_1(x,\xi,t-\tau)\,d\xi. \qquad (3,3_3)$$

Differentiating $(3,3_2)$ and using (2,10) and (3,2), we obtain

$$v_1(x,t) = \varphi(l)g_2^*(x,l,t) + I_{0,1}(x,t) + I_{0,2}(x,t \mid y,w), \qquad (3,4)$$

where

$$I_{0,1} = \int_l^\infty \dot\varphi(\xi)g_2^*(x,\xi,t)\,d\xi; \qquad I_{0,2} = b_2^2\int_0^t w(\tau)\frac{\partial}{\partial \xi}g_2^*(x,y(\tau),t-\tau)\,d\tau. \quad (3,5)$$

* g_1^* is easily found by using the Laplace–Carson transform. In the region $-\infty < x < \infty;\ 1 < \xi < \infty$, the construction of g_1^* is given in (4).

Equality $(3, 3_2)$ determines v for $x > y(t)$. At the same time $v_1(x,t)$ is defined only for $x > y(t)$. We continue v_1 to the half-line $x \le y(t)$ by means of the equalities

$$v_1(y(t),t) = \varphi(l)g_2^*(y(t),l,t) + \frac{1}{2}w(t) + I_{0,1}(y(t),t) + \\ + I_{0,2}(y(t),t \mid y,w); \qquad (3,5_1)$$

$$v_1(x,t) = \varphi(l)g_2^*(x,l,t) + w(t) + I_{0,1}(x,t) + I_{0,2}(x,t \mid y,w) \\ \text{for } x < y(t). \qquad (3,5_2)$$

By virtue of the theorem on the jumps of the thermal potential of a double layer and of the definition of g_1^* and g_2^*, the function $v_1(x,t)$ so defined will be continuous at $x = y(t)$, if $w(t)$ is continuous.

We shall write

$$2I_{0,1}(y(t),t) = I_{1,1}(t \mid y); \qquad 2I_{0,2}(y(t),t \mid y,w) = I_{1,2}(t \mid y,w). \qquad (3,6)$$

Then from $(3,1)$, $(3,2)$, and $(3,5_1)$ it follows that

$$w(t) = 2\varphi(l)g_2^*(y(t),l,t) + I_{1,1}(t \mid y) + I_{1,2}(t \mid y,w). \qquad (3,7)$$

Analogously we find

$$z_1(t) = 2\{[\psi_1(0) - f(0)]G_2(y(t),0,t) - \psi_1(l)G_2(y(t),l,t) + \\ + I_{2,1}(t \mid y) + I_{2,2}(t \mid y,z_1)\}, \qquad (3,8)$$

$$z(t) = 2\{\psi(l)g_2(y(t),l,t) + I_{3,1}(t \mid y) + I_{3,2}(t \mid y,z) + I_{3,3}(t \mid y,v_1)\}. \qquad (3,9)$$

Here

$$I_{2,1} = \int_0^l \psi_1(\xi)G_2(y(t),\xi,t)\,d\xi,$$

$$I_{2,2} = -\int_0^t z_1(\tau)\frac{\partial}{\partial\xi}G_2(y(t),y(\tau),t-\tau)\,d\tau, \qquad (3,10)$$

$$I_{3,1} = \int_l^\infty \psi(\xi) g_2(y(t), \xi, t)\, d\xi;$$

$$I_{3,2} = a_2^2 \int_0^t z(\tau) \frac{\partial}{\partial \xi} G_2(y(t), y(\tau), t-\tau)\, d\tau;$$

$$I_{3,3} = -\int_0^t d\tau \int_{y(\tau)}^\infty \gamma^2(\xi) v_1^2(\xi, \tau) \frac{\partial}{\partial \xi} g_2(y(t), \xi, t-\tau)\, d\xi.$$

To the system $(3,4)$–$(3,10)$ we adjoin condition $(1,3)$, which, by virtue of $(3,1)$, is written in the form

$$y(t) = l + \int_0^t [k_1 z_1(\tau) - k z(\tau)]\, d\tau. \tag{3,11}$$

This completes the required reduction.

The theorem of existence and uniqueness is proved locally under the assumption that $f(t)$ is twice differentiable, while $\psi(x)$ and $\psi_1(x)$ are three times differentiable, with $\psi_1(0) = f(0)$; $\varphi(l) = \psi(l) = \psi_1(l) = 0$; $0 < l < 1$. The equivalence theorem is proved in the same way as in ([2]).

Computing Center
of the Latvian State University
named after Peter Stuchka

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CITED LITERATURE

[1] S. S. Grigoryan, *Prikl. matem. i mekh.*, **22**, no. 5 (1958).
[2] L. I. Rubinshtein, *On Certain Nonlinear Problems Generated by the Fourier Equation*, Dissertation, Moscow State University, 1957.
[3] L. I. Rubinshtein, *DAN*, **142**, no. 3 (1962).
[4] B. M. Budak, A. A. Samarskii, A. N. Tikhonov, *Collection of Problems in Mathematical Physics*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.