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Abstract

Full Text

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ON THE HOMOTOPY THEORY OF FUNCTORS IN THE CATEGORY OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov on 21 X 1961)

The present note is devoted to a justification of the Eckmann-Hilton duality. In ⁽¹⁾, for every functor in the category of topological spaces with base point, a dual functor was defined and examples of dual functors were indicated. Here it will be shown that, analogously to the homotopy theory of topological spaces, a homotopy theory of functors can be constructed: the notions of a map, homotopy, fibration and cofibration of functors, homotopy and cohomology groups of functors, the Whitehead product in homotopy groups and the Kolmogorov-Aleksandrov product in groups of cohomologies of functors will be defined. Thus, functors in the category of topological spaces may be regarded as generalized topological spaces. In the homotopy theory of functors there exists an exact duality, generated by the duality constructed in note ⁽¹⁾. From the theorems of the homotopy theory of functors one can obtain, in various ways, theorems of the homotopy theory of spaces; one of these ways is based on the fact that the homotopy groups and the cohomology groups of a space A are closely related to the homotopy groups and the groups of cohomologies of the functor Σ_A (see ⁽¹⁾, p. 819). Theorems dual in the sense of Eckmann-Hilton are those theorems of the homotopy theory of spaces which are obtained from mutually dual theorems of the homotopy theory of functors. We shall use the terminology, notation, and definitions of note ⁽¹⁾, with the sole difference that all spaces will be assumed to be completely regular.

The foundation of the homotopy theory of functors is the notion of a map of functors ⁽¹⁾. Maps f and g of a functor T into a functor U are called homotopic if, for $0 \leq t \leq 1$, a map f_t of the functor T into the functor U is defined, continuously depending on t (the topology in the set of maps of one functor into another was introduced in ⁽¹⁾, p. 819), and moreover $f_0 = f$ and $f_1 = g$. The notions of homotopy, homotopy equivalence, etc., are defined analogously.

Fibrations and cofibrations of functors. A map p of a functor E into a functor B is called a **fibration** if the following axiom is satisfied. Let Z be any functor, $F : Z \rightarrow E$ a map; $f_t : Z \rightarrow B$ ($0 \leq t \leq 1$) a homotopy such that $pF = f_0$. Then there exists a homotopy $F_t : Z \rightarrow E$ ($0 \leq t \leq 1$) such that $F_0 = F$ and $pF_t = f_t$ ($0 \leq t \leq 1$). The map p is called an **R -fibration** if the functors E and B are reflexive ⁽¹⁾ and the homotopy lifting axiom is satisfied

for every reflexive functor Z .

A map i of a functor B into a functor E is called a **cofibration** if the following axiom is satisfied. Let Z be any functor, $F : E \rightarrow Z$ a map; $f_t : B \rightarrow Z$ ($0 \leq t \leq 1$) a homotopy such that $Fi = f_0$. Then there exists a homotopy $F_t : E \rightarrow Z$ ($0 \leq t \leq 1$) such that $F_0 = F$ and $F_t i = f_t$ ($0 \leq t \leq 1$). The map i is called an **R -cofibration** if the functors B and E are reflexive and the preceding axiom is satisfied for every reflexive functor Z .

Every R -fibration is a fibration. If $p : E \rightarrow B$ is a fibration of functors, then for every space X the mapping $p_X : EX \rightarrow BX$ is a fibration in the sense of Serre. If $i : B \rightarrow E$ is a cofibration, then for any space X the mapping $i_X : BX \rightarrow EX$ is a cofibration. Whether a similar assertion is true for R -cofibrations is unknown to us. *If $p : E \rightarrow B$ is a fibration, then $Dp : DB \rightarrow DE$ ⁽¹⁾ is an R -cofibration. If $i : B \rightarrow E$ is a cofibration or an R -cofibration, then $Di : DE \rightarrow DB$ is an R -fibration and, consequently, also a fibration.*

Let $p : E \rightarrow B$ be a fibration. A functor F is called its **fiber** if there exists a mapping $i : F \rightarrow E$ such that $p \circ i = 0$, and the following axiom is satisfied. For every reflexive functor Z and mapping $\alpha : Z \rightarrow E$ such that $p \circ \alpha = 0$, there exists one and only one mapping $\beta : Z \rightarrow F$ such that $\alpha = i\beta$. Let $i : B \rightarrow E$ be a cofibration or an R -cofibration. A functor Φ is called its **cofiber** if there exists a mapping $p : E \rightarrow \Phi$ such that $p \circ i = 0$, and the following axiom is satisfied. For every reflexive functor Z and mapping $\alpha : E \rightarrow Z$ such that $i \circ \alpha = 0$, there exists one and only one mapping $\beta : \Phi \rightarrow Z$ such that $\alpha = \beta \circ p$.

Any two reflexive fibers of a fibration coincide with one another. An R -fibration has a unique fiber, and this fiber is reflexive. As was noted above, for any space X the mapping $p_X : EX \rightarrow BX$ is a fibration. Denoting its fiber by FX , we obtain a functor F , which, as is easy to see, is the fiber of the fibration of functors $p : E \rightarrow B$. *If p is an R -fibration, then F , thus, is a reflexive functor. Any two reflexive cofibers of a cofibration or an R -cofibration coincide with one another. Every R -cofibration has a reflexive cofiber.*

It is constructed as the functor dual to the fiber of the dual fibration. If Φ is the cofiber of some cofibration or R -cofibration, then $DD\Phi$ is also the cofiber of this cofibration or R -cofibration. If the cofibration or R -cofibration $i : B \rightarrow E$ has the property that for every space X the mapping $i_X : BX \rightarrow EX$ is a cofibration of spaces, then the cofiber ΦX of this cofibration defines a functor Φ , which turns out to be the cofiber of i . *If $i : B \rightarrow E$ is a cofibration or R -cofibration, Φ is its cofiber, then $Di : DE \rightarrow DB$ is an R -fibration with fiber $D\Phi$, and if $p : E \rightarrow B$ is a fibration with reflexive fiber F , then $Dp : DB \rightarrow DE$ is an R -cofibration with cofiber DF . Thus, if $i : B \rightarrow E$ is an R -cofibration such that $i_X : BX \rightarrow EX$ is a cofibration for all spaces X , and ΦX is the cofiber of the cofibration i_X , then $Di : DE \rightarrow DB$ is a fibration with fiber $D\Phi$. In particular, for any Y the mapping $Di_Y : DEY \rightarrow DBY$ is a fibration in the sense of Serre with fiber $D\Phi Y$.*

Let $\alpha : T \rightarrow U$ be a mapping of functors, and X a space. Using Serre' s

construction (see ⁽²⁾, p. 200), construct an equivalent mapping $\alpha_X : TX \rightarrow UX$, a fibration whose space we denote by $T_\alpha X$, with projection $p(\alpha)_X$. The spaces $T_\alpha X$ and the mappings $p(\alpha)_X$ define a functor T_α and a mapping of functors $p(\alpha) : T_\alpha \rightarrow U$, which is a fibration of functors homotopy equivalent to the mapping α . By $U_\alpha X$ we denote the cylinder of the mapping α_X , and by $i(\alpha)_X$ the inclusion of the base of the cylinder, homeomorphic to TX . By U_α and $i(\alpha)$ we denote the functor and the mapping defined by the spaces $U_\alpha X$ and the mappings $i(\alpha)_X$. The mapping $i(\alpha)$ is a cofibration and an R -cofibration, homotopy equivalent to α . Let $D\alpha : DU \rightarrow DT$ be the mapping dual to α . The following assertions hold.

1. $DU_\alpha = (DU)_\alpha$ and $D[i(\alpha)] = p(D\alpha)$.
 2. If T and U are reflexive functors, then $DT_\alpha = (DT)_\alpha$ and $D[p(\alpha)] = i(D\alpha)$.
- The first of these assertions is proved directly; the second can be derived from the following theorem. If U is a reflexive functor and T is any functor, then $D(T/U) = DT \circ DU$. The proof of the theorem is rather complicated.

Homotopy and cohomology groups.

By Σ , Σ^i , Ω , and Ω^i we shall denote the functors which assign to each space, respectively, its suspension, its i -fold suspension, the space of its loops, and the i -fold space of its loops. If T and U are functors, then by $\{T \rightarrow U\}'$ we shall denote the set of homotopy classes of mappings T into U , i.e., the set of linearly connected components of the space $\{T \rightarrow U\}$. Let T and U be arbitrary functors. The set $\{T \rightarrow \Omega^i U\}' = \{\Sigma^i T \rightarrow U\}'$, for $i \geq 1$, is in the natural sense a group, commutative for $i \geq 2$. By the i -dimensional homotopy group, respectively cohomology group, of the functor T with coefficients in the functor U we shall mean the group $\{\Sigma^i U \rightarrow T\}$, respectively the group $\{T \rightarrow \Omega^i U\}$. These groups are denoted respectively by $\pi_i(T, U)$ and $H^i(T, U)$.

Let us note some simple relations between the homotopy and cohomology groups of functors and spaces. Obviously, $\pi_i(T, U) = H^i(U, T)$. Further, $\pi_i(\Sigma_A, 1) = \pi_i(A)$, where 1 means the identity functor. Thus the homotopy groups of spaces are a special case of the homotopy groups of functors. There is also

$$H^i(\Sigma_A, \Sigma_K(G, n)) = H^{n-i}(A, G)$$

for any space A and any abelian group G , i.e. the cohomology groups of spaces are a special case of the cohomology groups of functors. If the functor S is reflexive, then

$$\pi_i(S, T) = H^i(DS, DT)$$

for any functor T , and if the functor S is reflexive, then

$$H^i(S, T) = \pi_i(DS, DT)$$

for any functor S . These equalities mean duality between homotopy and cohomology groups.

Let $p : E \rightarrow B$ be a fibration with fiber F . Then for any functor T there are exact sequences:

$$\dots \rightarrow \pi_i(E, T) \rightarrow \pi_i(B, T) \rightarrow \pi_{i-1}(F, T) \rightarrow \pi_{i-1}(E, T) \rightarrow \dots \quad (1)$$

$$\dots \rightarrow H^i(T, E) \rightarrow H^i(T, B) \rightarrow H^{i-1}(T, F) \rightarrow H^{i-1}(T, E) \rightarrow \dots \quad (2)$$

which pass into one another under substitution according to the formula $\pi_i(S, T) = H^i(T, S)$. Let $i : B \rightarrow E$ be a cofibration, respectively an R -cofibration, with cofiber Φ . Then for any, respectively for any reflexive, functor T there are exact sequences

$$\dots \rightarrow H^i(E, T) \rightarrow H^i(B, T) \rightarrow H^{i-1}(\Phi, T) \rightarrow H^{i-1}(E, T) \rightarrow \dots \quad (3)$$

$$\dots \rightarrow \pi_i(T, E) \rightarrow \pi_i(T, B) \rightarrow \pi_{i-1}(T, \Phi) \rightarrow \pi_{i-1}(T, E) \rightarrow \dots \quad (4)$$

which pass into one another under substitution according to the formula $\pi_i(S, T) = H^i(T, S)$. The sequences (1) and (2) pass into the sequences (3) and (4) under substitution according to the formulas $\pi_i(S, T) = H^i(DS, DT)$ and $H^i(S, T) = \pi_i(DS, DT)$. Special cases of the sequences (1)–(4) are the exact homotopy sequence of a fibration, the exact cohomology sequence of a cofibration, and the exact sequences of homotopy and cohomology groups of spaces induced by an exact sequence of coefficients:

$$0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0.$$

Multiplicative structure of the homotopy and cohomology groups of functors.

Let $\alpha \in \pi_i(S, T)$, $\beta \in \pi_j(S, U)$. We define the product

$$\alpha \cdot \beta \in \pi_{i+j-1}(S, T \times U),$$

where by $T \times U$ is meant the functor which assigns to each space X the space $TX \times UX$. By definition α is a linearly connected component of the space $\{\Sigma^i T \rightarrow S\}$, β is a linearly connected component of the space $\{\Sigma^j U \rightarrow S\}$. Together they define a linearly connected component of the space

$$\{\Sigma^i T \rightarrow S\} \times \{\Sigma^j U \rightarrow S\} = \{\Sigma^i T \vee \Sigma^j U \rightarrow S\}.$$

The element $\alpha \cdot \beta$ is defined with the aid of the mapping

$$\Sigma^{i+j-1}(T \times U) \rightarrow \Sigma^i T \vee \Sigma^j U,$$

which is constructed in the following way. Let A and B be spaces. The composite mapping of spaces

$$\Sigma^{i+j-1}(A \times B) = S^{i+j-1} \# (A \times B) \rightarrow (S^i \vee S^j) \# A \times B = (S^i \# A \times B) \vee (S^j \# A \times B) \rightarrow$$

$$\rightarrow S^i \# A \vee S^j \# B = \Sigma^i A \vee \Sigma^j B,$$

where the mapping

$$S^{i+j-1} \# A \times B \rightarrow (S^i \vee S^j) \# A \times B$$

constructed with the aid of the well-known Whitehead map

$$S^{i+j-1} \rightarrow S^i \vee S^j$$

defines the required mapping of functors. Let $a \in H^i(S, T)$, $b \in H^j(S, U)$. Define the product

$$a \cdot b \in H^{i+j-1}(S, T \vee U),$$

where by $T \vee U$ is meant the functor that assigns to each space X the space $TX \vee UX$. This product is defined by means of the mapping of functors

$$\Omega^i T \times \Omega^j U \rightarrow \Omega^{i+j-1}(T \vee U),$$

which is constructed as follows. A point of the space $\Omega^i TA \times \Omega^j UA$ is a pair of mappings

$$S^i \rightarrow TA \quad \text{and} \quad S^j \rightarrow UA,$$

i.e., a mapping

$$S^i \vee S^j \rightarrow TA \vee UA,$$

which, when composed with the Whitehead mapping, gives a mapping

$$S^{i+j-1} \rightarrow TA \vee UA,$$

i.e., an element of $\Omega^{i+j-1}(T \vee U)A$.

Note that if T and U are reflexive functors, then the constructed mappings

$$\Sigma^{i+j-1}(T \times U) \rightarrow \Sigma^i T \vee \Sigma^j U$$

and

$$\Omega^i T \times \Omega^j U \rightarrow \Omega^{i+j-1}(T \vee U)$$

are dual to one another. Hence it is easy to obtain that the constructed multiplications pass into one another under the substitutions according to the formulas

$$\pi_i(S, T) = H^i(DS, DT)$$

and

$$H^i(S, T) = \pi_i(DS, DT)$$

(As can be proved,

$$D(T \vee U) = DT \times DU$$

and

$$D(T \times U) = DT \vee DU.)$$

We shall show that the Whitehead multiplication in homotopy groups and the Kolmogorov-Alexander multiplication in cohomology groups are special cases of the multiplications introduced. Indeed, if $S = \Sigma_A$, $T = U = 1$, then

$$\begin{aligned}\pi_i(S, T) &= \pi_i(A), & \pi_j(S, T) &= \pi_j(A), \\ \pi_{i+j-1}(S, T \times U) &= \pi_{i+j-1}(A \vee A),\end{aligned}$$

and it is easy to verify that the multiplication

$$\pi_i(S, T) \otimes \pi_j(S, U) \rightarrow \pi_{i+j-1}(S, T \times U)$$

coincides with the Whitehead multiplication

$$\pi_i(A) \otimes \pi_j(A) \rightarrow \pi_{i+j-1}(A \vee A).$$

If G is a ring, $S = \Sigma_A$, $T = U = \Sigma_{K(G, n)}$, then

$$\begin{aligned}H^i(S, T) &= H^{n-i}(A, G), \\ H^j(S, U) &= H^{n-j}(A, G).\end{aligned}$$

The group $H^{i+j-1}(S, T \vee U)$ is the set of connected components of the space

$$\begin{aligned}\{\Sigma^{i+j-1}\Sigma_A \rightarrow \Sigma_{K(G, n)} \vee \Sigma_{K(G, n)}\} &= \{\Sigma^{i+j-1}\Sigma_A \rightarrow \Sigma_{K(G, n)} \vee K(G, n)\} \\ &= \Omega_{\Sigma^{i+j-1}A}\{K(G, n) \vee K(G, n)\} = \Omega_{\Sigma^{i+j-2}A}\Omega(K(G, n) \vee K(G, n)).\end{aligned}$$

Let X and Y be two spaces, $\alpha \in H^i(X, G)$, $\beta \in H^j(X, G)$. By $\omega(\alpha, \beta)$ denote the element of the group

$$H^{i+j-2}(\Omega(X \vee Y); G)$$

constructed as follows. Let E be the Serre fibration of the space $X \vee Y$; let E_1 and E_2 be its restrictions respectively over X and Y . Clearly,

$$E_1 \cup E_2 = E, \quad E_1 \cap E_2 = \Omega(X \vee Y).$$

By

$$\alpha' \in H^{i-1}(E_2, G)$$

denote the image of α under the composite mapping

$$H^i(X; G) \rightarrow H^i(E_1, \Omega(X, Y)) \rightarrow H^i(E, E_2; G) = H^{i-1}(E_2, G).$$

Analogously one constructs

$$\beta' \in H^{j-1}(E; G).$$

Their product in

$$H^{i+j-2}(E_1 \cap E_2; G) = H^{i+j-2}(\Omega(X \vee Y); G)$$

is denoted by $\omega(\alpha, \beta)$. The mapping

$$H^{i+j-1}(S, T \vee U) \rightarrow H^{2n-i-j}(A; G)$$

is constructed as assigning to each element of $H^{i+j-1}(S, T \vee U)$, which, by the preceding, can be identified with a homotopy class of mappings of the space

$$\Sigma^{i+j-2} A$$

into

$$\Omega(K(G, n) \vee K(G, n)),$$

the image of $\omega(e, e)$ under the adjoint homomorphism of cohomology, where

$$e \in H^n(K(G, n); G)$$

is the fundamental class. The resulting mapping

$$H^{n-i}(A, G) \otimes H^{n-j}(A, G) \rightarrow H^{2n-i-j}(A, G),$$

as can be verified, coincides with the Kolmogorov-Alexander multiplication.

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CITED LITERATURE

1. D. B. Fuks, DAN, **141**, No. 4 (1961).
2. J. P. Serre, Comm. Math. Helv., **27**, 198 (1953).

Note: Figure translations are in progress. See original paper for figures.

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