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Abstract

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MATHEMATICS

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ON THE SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR THE POISSON EQUATION IN A RECTANGLE

(Presented by Academician A. A. Dorodnitsyn, 19 V 1962)

A theorem of S. M. Nikol'skii⁽¹⁾ is generalized; it establishes necessary and sufficient conditions for the solution of the Dirichlet problem for the Laplace equation in a rectangle D to belong to the class $C_{k,\lambda}(\overline{D})^*$ for the case of the first, second, and mixed boundary-value problems for the Poisson equation. When the matching conditions for the boundary functions at the corners are not fulfilled, a method is indicated for separating out the irregular component of the solution. Difference schemes are given for the grid method, making it possible to compute the regular solution of the second and mixed boundary-value problems for the Poisson equation with errors of order $O(h^{6-\varepsilon})$ and $O(h^6)$, respectively, in contrast to $O(h^{2-\varepsilon})$ and $O(h^2)$ for the known difference schemes^(2, 3). The method proposed by Bachelet⁽⁴⁾ for obtaining estimates of the rate of convergence of the grid method is refined.

1. Denote by D the rectangle $\{0 < x < a, 0 < y < b\}$. Number the sides of the rectangle Γ_j ($j = 1, 2, \dots$) counterclockwise, beginning with the side $x = 0$. Take the point $(0, b)$ as the origin for measuring the arc s of the boundary Γ , and denote by s_j the beginning of Γ_j . Suppose that in D the equation

$$\Delta u = f(x, y) \quad (1)$$

is prescribed, with boundary conditions

$$\nu_j u|_{\Gamma_j} + \bar{\nu}_j u_n|_{\Gamma_j} = \nu_j \varphi_j(s) + \bar{\nu}_j \psi_j(s), \quad j = 1, 2, 3, 4, \quad (2)$$

where u_n is the derivative along the interior normal; $\varphi_j(s)$, $\psi_j(s)$ are functions continuous on the segment $[s_j, s_{j+1}]$; ν_j may take the values 0 or 1; $\bar{\nu}_j = 1 - \nu_j$. Let $\sigma = \sum_{j=1}^4 \nu_j$. If $\sigma = 0$, then the condition

$$\sum_{j=1}^4 \int_{s_j}^{s_{j+1}} \psi_j(s) ds = - \iint_D f(x, y) dx dy$$

is satisfied. Suppose, moreover,

$$\nu_j \varphi_j(s) + \int_{s_j}^s \bar{\nu}_j \psi_j(s) ds \in C_{k,\lambda}([s_j, s_{j+1}]); \quad f(x, y) \in C_{k-2,\lambda}(\bar{D}), \quad (3)$$

$$k \geq 2, \quad 0 < \lambda < 1,$$

and, for $s = s_{j+1}$, the matching conditions hold

$$\begin{aligned} \chi_{jq} \equiv & \nu_{j+1} \varphi_{j+1}^{(2q+\delta_\tau-2)} + \bar{\nu}_{j+1} \psi_{j+1}^{(2q+\delta_\tau)} - \\ & - (-1)^{q+\delta_\tau+\delta_{\tau-1}} (\nu_j \varphi_j^{(2q+\delta_{\tau-1})} + \bar{\nu}_j \psi_j^{(2q+\delta_\tau)}) = \theta_{jq}, \end{aligned} \quad (4)$$

* $f \in C_{k,\lambda}(G)$, if f is continuously differentiable k times in G and all its k -th derivatives satisfy in G a Hölder

where $\tau = \nu_j + 2\nu_{j+1}$; $\delta_\omega = 1$ for $\omega = 0$; $\delta_\omega = 0$ for $\omega \neq 0$; $q = 0, 1, \dots, Q$;

$$Q = [(k - \delta_{\tau-1} - \delta_{\tau-2})/2] - \delta_\tau; \quad \theta_{j0} = 0;$$

$$\theta_{1q} = \sum_{\mu=0}^{q-1} (-1)^\mu \frac{\partial^{2q-1+\delta_\tau-\delta_{\tau-3}} f(0,0)}{\partial x^{2(q-\mu)-1-\delta_{\tau-1}-\delta_{\tau-3}} \partial y^{2\mu+\delta_\tau+\delta_{\tau-1}}}, \quad q > 0; \quad (5)$$

θ_{jq} ($j = 2, 3, 4$; $q > 0$) have expression (5) in a coordinate system with axes x and y passing through Γ_{j+1} and Γ_j .

Theorem 1. *In order that the solution of problem (1)–(2) satisfy $u \in C_{k,\lambda}(\bar{D})$, $k \geq 2$, $0 < \lambda < 1$, it is necessary and sufficient that conditions (3)–(4) be fulfilled.*

For $f \equiv 0$ and $\sigma = 4$, Theorem 1 belongs to S. M. Nikol'skii⁽¹⁾ and is a special case of V. V. Fufaev⁽⁵⁾. We shall prove the sufficiency of the conditions of Theorem 1 for $f \equiv 0$ and $\sigma < 4$.

$\sigma = 0$ (the Neumann problem). Compute

$$\varphi_1(s) = \int_0^s \psi_1(s) ds$$

and

$$\varphi_j(s) = \varphi_{j-1}(s_j) + \int_{s_j}^s \psi_j(s) ds, \quad j = 2, 3, 4.$$

Consider the solution of the equation $\Delta v = 0$ with $v|_{\Gamma_j} = \varphi_j$. From the fulfillment of (4) for ψ_j with $\nu_j = 0$ follows the fulfillment of (4) for φ_j with $\nu_j = 1$. Therefore $v \in C_{k,\lambda}(\overline{D})$. The function

$$u = \int_{z_0}^z -\frac{\partial v}{\partial y} dx + \frac{\partial v}{\partial x} dy + C,$$

where $z_0 = a/2 + ib/2$, $z = x + iy$, is the harmonic function conjugate to v ⁽⁶⁾, and, by virtue of the Cauchy-Riemann conditions, $u \in C_{k,\lambda}(\overline{D})$ and satisfies conditions (2) for $\sigma = 0$.

$\nu_1 = 0$, $\sigma = 3$. Extend the functions $\psi_j(s)$ ($j = 2, 3, 4$) so that for $\nu_j = 0$ conditions (3)–(4) are fulfilled and there exists a solution of the equation $\Delta v = 0$ with $v'_n|_{\Gamma_j} = \psi_j(s)$. According to the preceding, $v \in C_{k,\lambda}(\overline{D})$. Denote $w = u - v$. Obviously, $w'_n|_{\Gamma_1} = \Psi_1(s) \equiv 0$, and, since $v \in C_{k,\lambda}(\overline{D})$ and is harmonic in D , conditions (4) with $j = 2, 3$ are fulfilled for the functions $\Phi_j(s) = \varphi_j(s) - v|_{\Gamma_j}$ ($j = 2, 3, 4$), and moreover

$$\Phi_2^{(2q+1)}(b+0) = \Phi_4^{(2q+1)}(2a+2b-0) = 0$$

$$(q = 0, 1, \dots, ((k-1)/2)).$$

Construct the rectangle $D^* \{-a < x < a, 0 < y < b\}$. Continue, onto the part of the boundary Γ^* of the domain D^* situated to the left of the y -axis, the functions Φ_j symmetrically with respect to the y -axis, and obtain boundary conditions of the first kind for the equation Δw^* in D^* , satisfying conditions (3)–(4). The function $w^* \in C_{k,\lambda}(\overline{D}^*)$ and, being even with respect to the y -axis, coincides with w in \overline{D} . Consequently, $u = v + w \in C_{k,\lambda}(\overline{D})$.

$\nu_1 = \nu_3 = 0$, $\sigma = 2$. We construct a harmonic function $v \in C_{k,\lambda}(\overline{D})$ such that $w'_n|_{\Gamma_j} = 0$ ($j = 1, 3$). We continue the boundary conditions for w from Γ_j ($j = 2, 3, 4$) symmetrically with respect to the y -axis onto the part Γ^* situated to the left of the y -axis. The function w^* , which is a solution of the Laplace equation in D^* with the constructed mixed boundary conditions, according to ^(6,7) is bounded in \overline{D}^* and $w^* \in C_{k,\lambda}(\overline{D}')$, where D' is the rectangle $\{0 < x < 0.7a, 0 < y < b\}$. Since $w^* = w$ in D , $u = v + w \in C_{k,\lambda}(\overline{D}')$. By symmetry considerations $u \in C_{k,\lambda}(\overline{D}'')$, where D'' is the rectangle $\{0.3a < x < a, 0 < y < b\}$ and, consequently, $u \in C_{k,\lambda}(\overline{D})$.

In the remaining cases, for $f \equiv 0$, the proof of Theorem 1 is carried out analogously to the cases considered.

For $f \neq 0$ we extend it, according to Whitney and Hestenes ⁽⁸⁾, to a minimal circle E containing D . The extended function $f^* \in C_{k-2,\lambda}(\overline{E})$ and coincides with f in \overline{D} . According to ⁽⁹⁾, the solution of the equation $\Delta v = f^*$ in E with zero boundary values $v \in C_{k,\lambda}(\overline{E})$. We reduce the solution of problem (1)–(2) to finding a har-

monic function $w = u - v$. By direct calculations, using (1), we verify that for the functions $\varphi_j^* = \varphi_j - v|_{\Gamma_j}$, $\psi_j^* = \psi_j - v'_n|_{\Gamma_j}$ ($j = 1, 2, 3, 4$), (3)–(4) are satisfied. Consequently, the solution of problem (1)–(2) is $u = v + w \in C_{k,\lambda}(\overline{D})$.

The necessity of the conditions of Theorem 1 is quite obvious. If $u \in C_{k,\lambda}(\overline{D})$, then, in particular, (3) is satisfied. The fulfillment of conditions (4) is checked by differentiating (1).*

2. Theorem 2. *If (3) holds, but (4) is not fulfilled, then the solution of problem (1)–(2) can be represented in the form $u = \sum_{j=1}^4 F_j + u^*$, where*

$$F_1 = \frac{2}{\pi} \sum_{q=0}^Q \frac{\chi_{1q} - \theta_{1q}}{(2q+1 + \delta_\tau - \delta_{\tau-3})!} \left((\delta_\tau + \delta_{\tau-1}) \operatorname{Re}\{z^{2q+1+\delta_\tau} \ln z\} - (\delta_{\tau-2} + \delta_{\tau-3}) \operatorname{Im}\{z^{2q+1-\delta_{\tau-3}} \ln z\} \right), \quad (6)$$

F_j ($j = 2, 3, 4$) have expression (6) in the coordinate system with axes x and y passing through Γ_{j+1} and Γ_j , with $\chi_{1q} - \theta_{1q}$ replaced by $\chi_{jq} - \theta_{jq}$; the function $u^* \in C_{k,\lambda}(\overline{D})$ is a solution of equation (1) with boundary conditions (2) when φ_j and ψ_j are replaced by

$$\varphi_j^* = \varphi_j - \sum_{p=1}^4 F_p|_{\Gamma_j}, \quad \text{and} \quad \psi_j^* = \psi_j - \left(\sum_{p=1}^4 F_p \right)'_n \Big|_{\Gamma_j}.$$

By direct calculations we verify that for the functions φ_j^* and ψ_j^* , (3)–(4) are satisfied. On the basis of Theorem 1, $u^* \in C_{k,\lambda}(\overline{D})$.

3. For rational a/b it is possible to construct effective difference schemes for the approximate computation of the regular solution of problem (1)–(2). At grid nodes lying in D , for example at 7 (see Fig. 1), we use equation (10)

$$u_7 = \frac{u_3 + u_6 + u_{10} + u_8}{5} + \frac{u_2 + u_9 + u_{11} + u_4}{20} - \frac{3h^2}{10} f - \frac{3h^4}{4!5} \Delta f - \frac{3h^6}{6!5} \left(\Delta^2 f + 2f_{x^2 y^2}^{(4)} \right) + C_1 h^8 M_7^{(8)}; \quad (7)$$

at nodes lying inside Γ_j with $\nu_j = 0$, for example at 3, by the method of undetermined coefficients, using (1) and (2), we construct the equation

$$\begin{aligned}
 u_3 = & \frac{2}{5}u_7 + \frac{u_2 + u_4}{5} + \frac{u_6 + u_8}{10} - \frac{3h}{5}\psi_2 - \frac{3h^2}{10}f - \frac{h^3}{10}f_y^{(1)} \\
 & - \frac{3h^4}{4!5}\Delta f - \frac{h^5}{5!5}(3f_{y^3}^{(3)} + 7f_{x^2y}^{(3)} - 2\psi_2^{(4)}) - \frac{3h^6}{6!5}(\Delta^2 f + 2f_{x^2y^2}^{(4)}) \\
 & - \frac{h^7}{7!5}(3f_{y^5}^{(5)} + 18f_{x^2y^3}^{(5)} + 17f_{x^4y}^{(5)} - 10\psi_2^{(6)}) + C_2 h^8 M_3^{(8)} \quad (8)
 \end{aligned}$$

and at corner points for $\tau = 0$, for example at 1, the equation

$$\begin{aligned}
 u_1 = & \frac{2}{5}(u_2 + u_5) + \frac{u_6}{5} - \frac{3h}{5}(\psi_1 + \psi_2) - \frac{h^2}{5}\left(\frac{3}{2}f + \psi_2^{(1)}\right) \\
 & - \frac{h^3}{10}(f_x^{(1)} + f_y^{(1)}) - \frac{h^4}{4!5}(3\Delta f + 4f_{xy}^{(2)}) \\
 & - \frac{h^5}{5!5}(3f_{x^3}^{(3)} + 3f_{y^3}^{(3)} + 7f_{xy^2}^{(3)} + 7f_{x^2y}^{(3)} - 2\psi_1^{(4)} - 2\psi_2^{(4)}) \\
 & - \frac{h^6}{6!5}(3\Delta^2 f + 6f_{x^2y^2}^{(4)} + 10(\Delta f)_{xy}^{(2)} + 4\psi_1^{(5)} - 4\psi_2^{(5)}) \\
 & - \frac{h^7}{7!5}(3f_{x^5}^{(5)} + 3f_{y^5}^{(5)} + 18f_{x^3y^2}^{(5)} + 18f_{x^2y^3}^{(5)} + 17f_{x^4y}^{(5)} + 17f_{xy^4}^{(5)} \\
 & - 10\psi_1^{(6)} - 10\psi_2^{(6)}) + C_3 h^8 M_1^{(8)}, \quad (9)
 \end{aligned}$$

* **Note added in proof.** As became known to the author during the correction of Theorem 1, for $\sigma = -4$, $\varphi_j \equiv 0$ ($j = 1, 2, 3, 4$), $f \neq 0$, it was reported by V. V. Fufaev at a seminar at the V. A. Steklov Mathematical Institute of the USSR Academy of Sciences in 1961.

where C_α ($\alpha = 1, 2, \dots$), here and below, are constants independent of h and u ; $M_\beta^{(8)}$ are quantities not exceeding in absolute value the maximum of the absolute values of the eighth derivatives of u in \bar{D} ; the values of ψ_1, ψ_2 , and f , and of their derivatives, are taken at the point for which the equation is written.

Fig. 1

If equations (7)–(9) are written at all nodes of the grid in \bar{D} , except for the nodes at which u is known, and the terms with h^7 and h^8 are discarded, then we obtain a system of difference equations for the approximate solution \tilde{u} of problem (1)–(2). For $\sigma = 0$, in order to single out a unique solution, at one node closest to the center of D we fix \tilde{u} and discard the corresponding equation. The coefficients of the unknowns appearing on the right-hand side in (7)–(9)

are nonnegative and in sum do not exceed one, which ensures the validity of the maximum principle for the system of difference equations.

4. Theorem 3. If the solution of problem (1)–(2) satisfies $u \in C_{8,0}(\bar{D})$, then

$$|\tilde{u} - u| \leq C_4(1 + C_\varepsilon)(M^{(7)} + M^{(8)})h^{6-\varepsilon}\delta_\sigma,$$

where $\varepsilon > 0$; C_ε depends on ε ; $M^{(r)}$ is the maximum of the absolute values of the derivatives of u of order r in \bar{D} .

For $\sigma = 4$, Theorem 3 was proved by Sh. E. Mikeladze⁽¹¹⁾. For $\sigma = 0$, the method of proof is set forth in⁽¹²⁾. For $0 < \sigma < 4$, Theorem 3 can be proved by refining the method proposed by Batschelet⁽⁴⁾. In⁽⁴⁾, to majorize the error, the solution of the equation $\Delta w = -1$ with boundary conditions (2) for $\varphi_j(s) \equiv 1$ and $\psi_j(s) \equiv -1$ is used, under the assumption that $w \in C_{3,0}(\bar{D})$. But this assumption is not satisfied, since (4) is satisfied. However, if $\varphi_j(s)$ is modified so that (3) and (4) hold for $k = 3$, $\lambda > 0$, and the inequalities $1 \leq \varphi_j(s) \leq 2$ hold, then by means of the arguments of⁽⁴⁾ one can already obtain the inequality

$$|\tilde{u} - u| \leq C_5(M^{(7)} + M^{(8)})h^6$$

for $h \leq h_0$, where h_0 is a certain finite quantity. For $h > h_0$, the solution of the system of equations for the error is bounded by a constant multiplied by $(M^{(7)} + M^{(8)})$. Consequently, Theorem 3 is valid for any h realizable in \bar{D} .

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Note: Figure translations are in progress. See original paper for figures.

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