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Abstract

Full Text

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ON THE LIMITING BEHAVIOR OF COMPOSITIONS OF MEASURES ON THE COMPLEX UNIMODULAR GROUP

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The composition of two measures μ_1 and μ_2 (we consider only probability measures) defined on a topological group G is the measure $\mu_1 * \mu_2$ determined by the equality

$$\mu_1 * \mu_2(\Gamma) = \int_G \mu_2(a^{-1}\Gamma)\, \mu_1(da),$$

where Γ is any Borel set.

Let S be a class of certain Borel sets. We shall call the number

$$R_S(\mu, \nu) = \sup_{\Gamma \in S} |\mu(\Gamma) - \nu(\Gamma)|$$

the S-distance between the measures μ and ν (for example, if S is the class of all Borel sets, then R_S is the distance in variation). Suppose the class S is fixed. The question arises of the limiting behavior of the compositions

$$\mu^n = \mu * \cdots * \mu$$

(n times) as $n \to \infty$, i.e., first, under what conditions on the measures μ and ν, as $n \to \infty$, does $R_S(\mu^n, \nu^n) \to 0$ (in this case we shall say that μ^n and ν^n approach each other), and, second, how to give an estimate for the quantities $\mu^n(\Gamma)$, where $\Gamma \in S$, such that its error tends to zero as $n \to \infty$ uniformly in $\Gamma \in S$.

For the case when G is Euclidean space and S is the set of multidimensional intervals, these questions are solved by the central limit theorem (under the assumption that the distributions under consideration have second moments).

In papers ([3],[4]) the limiting behavior of compositions of measures on certain homogeneous spaces is considered. It is easy to see that the problem considered there is equivalent to the problem of compositions of symmetric measures on the corresponding groups (a symmetric measure is a measure μ having the property $\mu(u_1 \Gamma u_2) = \mu(\Gamma)$, where Γ is any Borel set, and u_1 and u_2 are arbitrary elements of a maximal compact subgroup). In this case the limiting behavior of the measure μ^n is determined, as in the Euclidean case, by a collection of a finite number of quantities playing the role of moments and constructed from the measure μ.

In the present paper (here G is the unimodular group) we abandon the assumption of symmetry of the measures (introducing instead various assumptions about the existence and properties of their densities). It is shown that if for the class S one takes the class of sets determined by conditions on the eigenvalues of matrices, then the limiting behavior of μ^n, as in the preceding cases, is determined by a finite collection of numbers constructed from the measure μ. If, however, for the class S one takes the collection of all Borel subsets of the unimodular group, then, in order that $R_S(\mu^n, \nu^n) \to 0$ as $n \to \infty$, it is necessary (the author does not know whether it is sufficient) that certain functions constructed from these measures coincide, i.e., the limiting behavior depends in this case on a point of an infinite-dimensional space.

1. Let us consider the complex unimodular group of second order A_1 (concerning the definitions of the concepts used below and partially the nota-

see ([1], [2])). Let the probability measure μ on the group A_1 have a continuous density $\chi(a)$ with respect to Haar measure da, and suppose that the integral $\int \chi(u^{-1}ku_1)\, d_l k$ converges uniformly in $u, u_1 \in U$ (the notation $d_l k$ for the left-invariant measure on the group K of matrices of the form $k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}$ indicates at the same time that the integral is taken over the subgroup K; U is the unitary subgroup). Let $a \to T_a$ be a representation of the principal series in functions on the group U, determined by the parameters m and ρ, where $\rho = \sigma - 2i$, σ is a real number. Then the operator $T_\chi = \int T_a \chi(a)\, da$ will be an integral operator on U with kernel

$$K_\chi(u, u_1, m, \rho) = \pi \int \chi(u^{-1}ku_1)\, \alpha(k)\, d_l k,$$

where $\alpha(a) = |a_{22}|^{i\rho - m - 2} a_{22}^m$, and $K_{\chi^n} = K_\chi^n$, where K_χ^n is the n-th iteration of the kernel K_χ (χ^n is the density of the measure μ^n).

Together with the density χ^n, consider $\bar{\chi}_n(a) = \iint \chi^n(u_1 a u_2)\, du\, du_1$. Since $\bar{\chi}_n(a) = \bar{\chi}_n(u_1 a u_2)$ for any $u_1, u_2 \in U$, $\bar{\chi}_n(a)$ may be regarded as the density of a symmetric measure $\tilde{\mu}_n$ (see ([3])) on the space $M = A_1/U$, the Lobachevsky space. Denote the eigenvalues of the matrix aa by $e^{2\eta}$ and $e^{-2\eta}$ (where $\eta \geq 0$). Then η may be regarded as the Riemannian distance $\rho(O, x)$ from the origin in M to the point x corresponding to the adjacent class containing a. Denote by $p_n(\eta)$ the derivative

$$\frac{d}{d\eta} \tilde{\mu}_n\{x : \rho(O, x) < \eta\}.$$

Let

$$\Gamma_\eta = \{a : a \in A_1,\ \varepsilon(a) + \varepsilon(a)^{-1} < 2\,\mathrm{ch}\,\eta\},$$

where $[\varepsilon(a)]^2$ and $[\varepsilon(a)]^{-2}$ are the eigenvalues of the matrix aa^*. Since $u_1\Gamma_\eta u_2 = \Gamma_\eta$ for any $u_1, u_2 \in U$, it follows that

$$\mu^n(\Gamma_\eta) = \int_{\Gamma_\eta} \bar{\chi}_n(a)\,da = \int_0^\eta p_n(\eta)\,d\eta.$$

Thus, if for the class S one takes $\{\Gamma_\eta,\ 0 \le \eta < \infty\}$, then it suffices to study the behavior of the measure $\tilde{\mu}_n$ in the space M. The characteristic function of the measure $\tilde{\mu}_n$* is defined by the equality

$$f_n(\sigma - 2i) = \int (T_a e, e)\bar{\chi}_n(a)\,da,$$

where T_a is the representation with $m = 0$ and $\rho = \sigma - 2i$; e is the function on the unitary group identically equal to one. It is easy to verify that

$$f_n(\sigma - 2i) = \iint K_\chi^n(0, \sigma - 2i, u, u_1)\,du\,du_1.$$

Thus, $f_n(\sigma - 2i)$ is a positive-definite function of σ. Denote its inverse Fourier transform by $q_n(y)$, and the measure with density $q_n(y)$ by m_n. It is easy to see that

$$q_n(y) = 2e^{2y} \int_{|y|}^\infty \frac{p_n(y)}{\mathrm{sh}\,2\eta}\,d\eta$$

(note that at the point $\eta = 0$ the function $p_n(\eta)$ has order η^2).

* See ([3]); it should be borne in mind that in ([3]) a different model of Lobachevsky space is used, and therefore the formulas are somewhat different.

Lemma 1. The following relation holds: as $n \to \infty$,

$$\sup_{0 \le \eta < \infty} \left| \int_0^\eta p_n(\eta)\,d\eta - \int_0^\eta q_n(y)\,dy \right| \to 0.$$

Thus it is sufficient to study the limiting behavior of the measure m_n defined on the line.

Lemma 2. For $m = 0$, $\sigma = 0$ (i.e., $\rho = -2i$), the eigenvalue $\lambda = 1$ of the kernel K_x is simple. For $|\sigma| \le \sigma_0$, where σ_0 is some positive number depending on x, the following equalities hold:

$$K_x(u, u_1, 0, \sigma - 2i) = \lambda_x(\sigma)\varphi_x(\sigma, u)\psi_x(\sigma, u_1) + L_x(u, u_1, 0, \sigma - 2i),$$

$$K_x^n(u, u_1, 0, \sigma - 2i) = \lambda_x^n(\sigma)\varphi_x(\sigma, u)\psi_x(\sigma, u_1) + L_x^n(u, u_1, 0, \sigma - 2i),$$

where $\lambda_x(0) = 1,\ \varphi_x(0, u) = 1,\ \int \psi_x(0, u_1)\, du_1 = 1$, and $\lambda_x(\sigma)$ is the eigenvalue of the kernel K_x of greatest modulus; moreover, uniformly for $|\sigma| \leq \sigma_0$,

$$\max_{u, u_1 \in U} |L_x^n(u, u_1, 0, \sigma - 2i)| \to 0 \quad \text{as } n \to \infty.$$

Now suppose additionally that

$$\int \ln^2 |\lambda|\, x(u^{-1} k u_1)\, d_1 k < B < \infty \tag{1}$$

(this condition is analogous to the condition for the existence of second moments).

Theorem 1. The sequence of measures m_n is asymptotically normal with parameters:

$$\frac{1}{i}\frac{d\lambda_x(\sigma)}{d\sigma}\bigg|_{\sigma=0}, \quad \left\{-\frac{d^2\lambda_x(\sigma)}{d\sigma^2}\bigg|_{\sigma=0} - \left[\frac{1}{i}\lambda_x'(0)\right]^2\right\}^{1/2}.$$

Corollary. If condition (1) is satisfied, the condition $\hat{\lambda}_x'(0) = \lambda_y'(0)$, $\lambda_x''(0) = \lambda_y''(0)$ is necessary and sufficient in order that, for measures μ and ν with densities x and y, the following relation hold: as $n \to \infty$,

$$\sup_{\Gamma_\eta,\ 0 \leq \eta < \infty} \left|\mu^n(\Gamma_\eta) - \nu^n(\Gamma_\eta)\right| \to 0.$$

2. The definitions and results of item 1 are easily transferred to the unimodular group of order n, A_{n-1}. By $\varepsilon(a)$ one should understand the set of square roots of the eigenvalues of the matrix aa^*, taken in decreasing order;

$$\Gamma_\eta = \{a : \varepsilon_k(a) \leq e^{\eta_k},\ k = 1, \dots, n\},$$

where $\eta = (\eta_1, \dots, \eta_n)$ is an arbitrary vector. The measure $\tilde{\mu}_n$ on the space of classes adjacent with respect to the unitary subgroup is defined in the same way as in item 1. By ρ and σ one should understand the vectors:

$$\sigma = (\sigma_1, \dots, \sigma_n), \quad \rho = \{\sigma_1 + (n-1)i,\ \sigma_2 + (n-3)i, \dots, \sigma_n - (n-1)i\},$$

and by the derivatives in Theorem 1 and its corollary—the corresponding partial derivatives at the point $\sigma = (0, \dots, 0)$. The measure m_n is defined as a measure in the Euclidean space $(y_1, \dots, y_n)$, whose Fourier transform is $f_n(\rho)$. With the indicated obvious changes of notation, Theorem 1 and its corollary remain valid.

3. Suppose now that the densities x_1 and x_2 of the measures μ_1 and μ_2 on the group A_1 satisfy the following conditions: for $i = 1, 2$,

$$\int |\ln|\lambda||^3 \, x_i(u^{-1} k u_1) \, d_1 k < A < \infty,$$

$$\int \left| \frac{\partial}{\partial|\lambda|} x_i(u^{-1} k u_1) \right| d_1 k < C < \infty.$$

Theorem 2. If, for measures with densities x_1, x_2, the conditions of the corollary to Theorem 1 are satisfied, then the variation distance between the measures $(\tilde{\mu}_1)_n$, $(\tilde{\mu}_2)_n$ tends to zero as $n \to \infty$.

Thus, if S is the smallest σ-algebra containing all Γ_η, $0 \le \eta < \infty$, then, under the conditions of the corollary to Theorem 1, $R_S(\mu_1^n, \mu_2^n) \to 0$ as $n \to \infty$.

4. Let us now consider the question of when the relation $R_S(\mu_1^n, \mu_2^n) \to 0$ as $n \to \infty$ holds, where S is the class of all Borel sets of the group A_1. We note that in the space of all continuous functions on the group U, with norm $\|\varphi(u)\| = \max_{u \in U} |\varphi(u)|$, the operators T_a for $m = 0$, $\rho = \sigma - 2i$ have norm not exceeding 1. Therefore, if $R_S(\mu_1^n, \mu_2^n) \to 0$, then

$$\left\| T_{x_1^n} - T_{x_2^n} \right\| = \sup_{\|\varphi\|=1} \left\| T_{x_1^n}\varphi - T_{x_2^n}\varphi \right\| =$$

$$= \sup_{\|\varphi\|=1} \left\| \int T_a \varphi (x_1^n(a) - x_2^n(a)) \, da \right\| \le \int |x_1^n(a) - x_2^n(a)| \, da \to 0.$$

The kernel of the operator $T_{x_i^n}$, $i = 1, 2$, has the form

$$K_{x_i^n}(0, \sigma - 2i, u, u_1) = \lambda_{x_i}^n \varphi_{x_i}(\sigma, u) \psi_{x_i}(\sigma, u_1) + L_{x_i}^n(0, \sigma - 2i, u, u_1),$$

where $\|L_{x_i}^n\| \to 0$ as $n \to \infty$.

Since $\lambda_{x_i}(0) = 1$, $\varphi_{x_i}(0, u) = 1$, a necessary condition for convergence in variation of measures with densities x_1^n and x_2^n is the equality $\psi_{x_1}(0, u_1) = \psi_{x_2}(0, u_1)$.

Thus, in this case the limiting behavior depends not on a finite set of numbers, but on the function $\psi_x(0, u)$. It is easy to prove that the set of functions $\{\psi_x\}$, corresponding, for example, to finitely infinitely differentiable x with norm $\|\psi_x\| = \max_{u \in U} |\psi_x(0, u)|$, cannot be mapped topologically onto a subset of a Euclidean space of finite dimension.

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Note: Figure translations are in progress. See original paper for figures.