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Abstract

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MATHEMATICS

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A STRENGTHENING OF THE LOZINSKY-KHARSHILADZE THEOREM ON POLYNOMIAL APPROXIMATIONS

(Presented by Academician S. N. Bernstein on 30 X 1961)

1. Let C be the space of all 2π-periodic complex-valued continuous functions $f(x)$, defined for $0 \leq x \leq 2\pi$, with norm

$$\|f\|_C = \max_{x \in [0,2\pi]} |f(x)|.$$

Consider a linear operator $U(f, x)$ mapping the space C into its subspace $\mathcal{E}_n$, formed by trigonometric polynomials

$$E_n = \sum_{|k| \leq n} c_k \exp(ikx)$$

of order $\leq n$, and let

$$U(\exp(ilx), x) = \sum_{|k| \leq n} \gamma_{l,k} \exp(ikx), \qquad l = 0, \pm 1, \pm 2, \ldots$$

The equalities $\gamma_{k,k} = 1$, $|k| \leq n$; $\gamma_{l,k} = 0$, $l \neq k$, are necessary and sufficient in order that the relations $U(E_n, x) \equiv E_n$ hold for each of the polynomials E_n of order $\leq n$. But we consider arbitrary operators U, imposing no restrictions whatever on the coefficients $\gamma_{l,k}$.

2. For the given operator $U(f, x)$ we construct the associated operator $U^0(f, x)$, setting, by definition,

$$U^0(f, x) = \int_0^{2\pi} f(x + t)\Phi(t)\, dt = \sum_{|k| \leq n} c_k(f)\gamma_{k,k} \exp(ikx),$$

where

$$\Phi(t) = (2\pi)^{-1} \sum_{|k| \leq n} \gamma_{k,k} \exp(-ikt)$$

and $c_k(f)$ are the Fourier coefficients of the function $f(x)$. By f_t we denote $f(x+t)$, where t is regarded as a parameter.

Theorem 1. If $f \in C$ and $U(f,x)$ is a linear operator mapping C into $\mathcal{E}_n$, then the identity

$$(2\pi)^{-1}\int_0^{2\pi} U(f_t, x-t)\,dt = \int_0^{2\pi} f(x+t)\Phi(t)\,dt = U^0(f,x). \tag{1}$$

holds.

Proof. Since the closure in the norm $\|f\|_C$ of the set of all trigonometric polynomials coincides with the space C, it is sufficient to prove identity (1) only for polynomials

$$f_N(x) = \sum_{|l|\le N} c_l(f_N)\exp(ilx)$$

of arbitrary order N. We have, putting $N > n$:

$$\int_0^{2\pi} U(f_{Nt}, x-t)\,dt = \int_0^{2\pi} U\left(\sum_{|l|\le n} c_l(f_N)\exp(il(x+t)), x-t\right) dt. \tag{2}$$

Equality (2) is equivalent to the content of Lemma 2 from the work (^{1}). Further:

$$\begin{aligned}\int_0^{2\pi} U\left(\sum_{|l|\le n} c_l(f_N)\exp(il(x+t)),\, x-t\right) dt &= \sum_{|l|\le n} c_l(f_N)\int_0^{2\pi}\left[\exp(ilt)\sum_{|k|\le n}\gamma_{l,k}\exp(ik(x-t))\right] dt\\ &= 2\pi\sum_{|l|\le n} c_l(f_N)\gamma_{l,l}\exp(ilx).\end{aligned} \tag{3}$$

Equalities (2) and (3) prove Theorem 1. Identity (1) is a generalization of the well-known Marcinkiewicz-Berman identity (^{1}), which is obtained from (1) under the special assumption that $\gamma_{l,k} = 0,\ k \ne l$. ($\Phi(t)$ can be identified with any polynomial from $\mathcal{E}_n$ by a suitable choice of the diagonal coefficients $\gamma_{k,k},\ |k| \le n$.)

3. Theorem 1 leads, in particular, to a lower estimate for the norm of an arbitrary operator U taking its values in $\mathcal{E}_n$:

$$\|U\|_C \ge \|U^0\|_C. \tag{4}$$

For the proof it suffices in identity (1) to put $x = 0$ and to consider the upper bounds of the moduli of both sides of this identity for all $f \in C$ satisfying $\|f\|_C \le 1$. If, for example, for the operator U the diagonal coefficients $\gamma_{k,k}$ satisfy the condition

$$\gamma_{k,k} = 1, \qquad |k| \leq n, \tag{5}$$

then, whatever the other coefficients $\gamma_{k,l},\ k \neq l$, may be, the estimate

$$\|U\|_C \geq 4\pi^{-2} \ln n + O(1), \quad n \to \infty,$$

is valid, since under condition (5) we have

$$\|U^0\|_C = 4\pi^{-2} \ln n + O(1), \quad n \to \infty.$$

4. The indicated estimate for the norm of the operator U makes it possible to strengthen somewhat the well-known theorem of Lozinskii-Kharshiladze (^{2,3}) on polynomial approximations of continuous functions.

Theorem 2. *Let $U_n(f,x)$ be linear operators mapping the space C into its subspaces $\mathcal{E}_n$, formed by trigonometric polynomials of order $\leq n$, $n = 1, 2, \ldots$. If, for the diagonal coefficients $\gamma_{k,k}$ of each of the operators U_n, condition (5) is satisfied, then, whatever the remaining coefficients $\gamma_{l,k}$, $l \neq k$, of these same operators may be, the limiting relations*

$$\|U_n(f,x) - f(x)\|_C \to \infty, \quad n \to \infty,$$

cannot hold for all $f \in C$.

In particular, if $\gamma_{k,l} = 0$ for all $k \neq l$, then Theorem 2 becomes the theorem of Lozinskii-Kharshiladze, if, of course, one disregards the inessential circumstance that the formulation of this theorem was given by its authors for the case of the space of **real** continuous functions C.

An analogous result is valid for approximations of continuous functions by algebraic polynomials.

For simplicity we have restricted ourselves to consideration of the classical space C of continuous periodic functions with norm

$$\|f\|_C = \max |f(x)|,$$

but Theorem 1 and inequality (4), together with the proofs, carry over almost literally to the case of a number of more general functional spaces, for example, spaces of type E, $\widetilde{F}_\theta$ from papers (^{4,5}). Finally, Theorem 2 can be strengthened a little further if, instead of condition (5), one requires the equalities $\gamma_{k,k} = 1$ to hold for $|k| \leq \nu_n$, where

$$\lim_{n \to \infty} \frac{\nu_n}{n} = 1.$$

The proof follows from the fact that under this condition the norms $\|U_n^0\|_C$ increase without bound (see, for example, (^{6}), p. 148).

5. Of substantial importance is the possibility of generalizing identity (1) to spaces of functions defined on topological groups. Let G be a bicompact commutative group with invariant measure μ ($\mu(G) = 1$), and let $\chi_k(x)$, $k = 1, 2, \ldots$, be the totality of its characters. Let $C(G)$ denote the normed space of all continuous functions $f(x)$ on the group G, whose norm is defined by the equality $\|f\|_C = \sup_{x \in G} |f(x)|$.

Let $U(f, x)$ be a linear operator mapping $C(G)$ into its subspace $X_n(G)$, formed by all polynomials $\sum_{k=1}^{n} c_k \chi_k(x)$ of order $\leq n$ (c_k are complex numbers). For any $f \in C(G)$, let $f_t = f(x+t)$, $c_k(f)$ be the Fourier coefficient with respect to the character $\chi_k(x)$, and

$$U(\chi_l, x) = \sum_{k=1}^{n} \gamma_{l,k} \chi_k(x).$$

Theorem 3. If $f \in C(G)$; $U(f, x)$ is a linear operator mapping $C(G)$ into the subspace $X_n(G)$, then

$$\int_G U(f_t, x - t)\, d\mu(t) = \sum_{k=1}^{n} c_k(f) \gamma_{k,k} \chi_k(x). \tag{6}$$

Identity (6) is a certain generalization of the identity established by D. L. Berman for bicompact groups [7], where again it is assumed that for the operator U the coefficients $\gamma_{l,k} = 0$ if $k \neq l$.

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