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Abstract

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On the Question of the Generalized Character of Commutation Functions

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It is known that the commutation function D of free-field operators is generalized and has the meaning of the kernel of functionals defined on the class of sufficiently smooth and rapidly decreasing functions g. However, in field theory one often has to introduce a certain "explicit form" of the D-function. It is usually written as follows:

$$D(x) = \frac{1}{(2\pi)^3 i} \int e^{ikx} \varepsilon(k^0) \delta(k^2 - m^2)\, dk. \tag{1}$$

Here and below we restrict ourselves to a real scalar field. The use of D-functions of the form (1) for calculating matrix elements is associated with well-known difficulties connected with the appearance of ultraviolet divergences. As was noted already by N. N. Bogolyubov in [1], in fact here we encounter not so much divergent expressions as undefined ones. This is due to the fact that expression (1) loses its meaning at $x = 0$ and therefore needs to be completed by a definition. Moreover, the product of D-functions needs an independent completion by definition. Carrying this out in one way or another, we may arrive at either finite or infinite expressions. The appearance of ultraviolet divergences under the usual method of work may be regarded as an "unsuccessful" way of completing the definition of the D-functions and their products, and renormalization as a correction of this method.

However, the appearance of meaningless divergent expressions at intermediate stages can in general be avoided if we abandon the "explicit" representation (1) of the D-function and replace it by a certain "limiting" one,

$$D(x) = \frac{1}{(2\pi)^3 i} \lim_{\mu \to 0} \Delta(x, \mu), \tag{2}$$

where $\Delta(x, \mu)$ for $\mu \neq 0$ is an ordinary function. Such a representation, evidently, corresponds more to the generalized character of the D-function than the representation (1). The "limiting" representation (2) and the rules of passage to the limit with respect to μ set forth below may be regarded as a "successful" way of completing the definition of generalized functions, since in this case we do not encounter any divergent expressions.

The function $\Delta(x,\mu)$ may be chosen in many different ways; we shall use for it an expression which, in our opinion, is the simplest. Let us represent $\Delta(x,\mu)$ in the form

$$\Delta(x,\mu) = \int dk\, e^{ikx} \sum_i C_i(\mu)\delta\big(k^2 - m^2\mathfrak{M}_i(\mu)\big)\,\varepsilon(k^0). \qquad (3)$$

For simplicity, we shall henceforth assume that $\mu \geq 0$. The representation of the D-functions written above is, evidently, equivalent to introducing auxiliary Pauli–Villars masses directly into the initial expression for the commutation functions.

If we require that the conditions

$$\sum_i C_i(\mu)\mathfrak{M}_i^{\alpha}(\mu) = 0, \qquad \alpha = 0,1,\dots,p, \qquad (4)$$

be satisfied, then $\Delta(x,\mu)$ will no longer be a generalized function, but an ordinary function of x, and the integration of $\Delta(x,\mu)$ with some test function $g(x)$ will not require an additional definition. It is therefore natural to define the integral of a $D(x)$-function by means of the equality

$$\int dx\, g(x)D(x) = \frac{1}{(2\pi)^3 i}\lim_{\mu\to 0}\int dx\, g(x)\Delta(x,\mu). \qquad (5)$$

Let us note that representation (2), together with assumptions (4) and (5), corresponds to the fact that at infinite momenta the field operators commute with one another. Since the integrals of individual commutator functions, multiplied by sufficiently smooth $g(x)$, have well-known values, the quantities C_i and $\mathfrak{M}_i$ must satisfy conditions ensuring that integrals of this type, calculated by means of (2), are equal to these values. Obviously, for this it is sufficient that the following conditions be fulfilled:

$$\begin{aligned}
&1)\quad \lim_{\mu\to 0} C_i(\mu) = C_i;\\
&2)\quad C_0 = 1;\\
&3)\quad \mathfrak{M}_i(\mu) < A(\varepsilon) \quad \text{for } \mu > \varepsilon;\\
&4)\quad \lim_{\mu\to 0}\mathfrak{M}_0(\mu) = 1;\\
&5)\quad \mathfrak{M}_{i\neq 0}(\mu) \to \infty \quad \text{as } \mu \to 0.
\end{aligned} \qquad (6)$$

Let us now turn to a consideration of the consequences of the generalized character of commutator functions for operators and state vectors. Let the state vectors describing the physical system form a Hilbert space $\mathcal{H}$. As is easy to see, the free-field operators $\varphi(x)$ are not proper operators of this space ([2]). Indeed, let $|\Omega\rangle$ describe the vacuum. Then the state $|\varphi(x)\rangle$, obtained by the action of $\varphi(x)$ on $|\Omega\rangle$, is evidently nonnormalizable, and therefore does not belong to the space $\mathcal{H}$. However, one may try to give the state $|\varphi(x)\rangle$ a meaning analogous to the meaning of a generalized function. For this purpose we form the functional state

$$|\Phi_g\rangle = \int dx\, g(x)|\varphi(x)\rangle. \tag{7}$$

For sufficiently smooth $g(x)$, the state $|\Phi_g\rangle$ has finite norm and belongs to $\mathcal{H}$. Thus we have the right to speak of $|\varphi(x)\rangle$ as a generalized state. Correspondingly, to the field operator $\varphi(x)$ one may assign the meaning of a generalized operator, i.e., define it as the kernel of a system of functional operators

$$\Phi_g = \int dx\, g(x)\varphi(x). \tag{8}$$

It follows from what has been said that one cannot assert that a definite value of the variable x corresponds to a definite value of the operator $\varphi(x)$ (and analogously for the state). However, as in the case of generalized functions, one has to introduce a conditional "explicit" dependence of the operator (state) on x. As before, this can be done correctly only by means of a limiting transition. In particular, $\varphi(x)$ can be represented in the form

$$\varphi(x) = \lim_{\mu\to 0} \varphi(x,\mu), \tag{9}$$

where the operator $\varphi(x,\mu)$ satisfies, for example, the commutation relation

$$[\varphi(x,\mu_1), \varphi(y,\mu_2)] = \frac{1}{(2\pi)^3 i}\,\Delta(x-y,\mu_1+\mu_2). \tag{10}$$

The operator $\varphi(x,\mu)$ transforms normalizable state vectors into normalizable ones, and one may already say of them that they are functions of x. Analogously one can specify an "explicit form" of a generalized state. In doing so, of course, it is assumed that a law is given for constructing the functional operator (state) Φ_g ($|\Phi_g\rangle$). Just as in the case of generalized functions, we shall put, by definition,

$$\int dx\, g(x)\varphi(x) = \lim_{\mu\to 0}\int dx\, g(x)\varphi(x,\mu). \tag{11}$$

The generalized character of operators and states must be taken into account when it is necessary to differentiate them or to pass to a limit, since these operations require an additional definition. Let us consider, as an example, the operation of differentiation. Let $|\Phi(x)\rangle$ be a generalized state, and let $|\Phi'(x)\rangle$ be the derivative of this state with respect to x, with $|\Phi'(x)\rangle$ also a generalized state. As already indicated, strictly speaking, only a functional state has meaning; in the present case it has the form

$$|\Phi'_g\rangle = \int dx\, g(x)|\Phi'(x)\rangle. \tag{12}$$

Writing $|\Phi'(x)\rangle$ formally as $|\Phi'(x)\rangle = \dfrac{\partial}{\partial x}|\Phi(x)\rangle$ and carrying out integration by parts, we obtain

$$|\Phi'_g\rangle = \int dx\,(-g'(x))|\Phi(x)\rangle. \tag{13}$$

It is natural to take equality (13) as the definition of differentiation of a generalized state. Rewriting (13), using (12), we have

$$|\Phi'_g\rangle = \lim_{\mu\to 0}\int dx\,(-g'(x))|\Phi(x,\mu)\rangle.$$

Since the integrand contains no generalized state, we can integrate it by parts:

$$|\Phi'_g\rangle = \lim_{\mu\to 0}\int dx\, g(x)\frac{\partial}{\partial x}|\Phi(x,\mu)\rangle, \tag{14}$$

or, in other words,

$$\frac{\partial}{\partial x}|\Phi(x)\rangle = \lim_{\mu\to 0}\frac{\partial}{\partial x}|\Phi(x,\mu)\rangle. \tag{15}$$

The last relation can also be taken as the definition of differentiation of a generalized state. Analogously one can verify that passage to the limit in a generalized state should be understood as passage to the limit under the sign $\lim_{\mu\to 0}$.

In addition to states of the type $|\varphi(x)\rangle$, in field theory one often has to deal with states which, although they are functionals, involve integration with sharply discontinuous functions g. These are the so-called states at a given instant of time or, in the more general case, states specified on some spacelike surface. Since the function g is not smooth in this case, such states will be generalized. This must be taken into account when we differentiate such a state or carry out in it a limiting transition. It must also be remembered that, in order to formulate the causality principle, we are forced to use

precisely such states; therefore all equalities that are consequences of the principle of causality should be understood in the generalized sense, i.e., generalized functions and operators should be represented in the form of the limiting expressions (2) and (9), and the expressions under the limit sign with respect to μ should be equated. In accordance with this, the chronological convolution of two fields should be introduced in the following way:

$$D^c(x) = \theta(x^0)D^-(x) - \theta(-x^0)D^+(x) =$$
$$= \frac{1}{(2\pi)^4} \lim_{\mu\to 0} \int dk\, e^{ikx} \sum_i C_i(\mu) \frac{1}{m^2\mathfrak{M}_i(\mu) - k^2 - i\varepsilon}. \tag{16}$$

In brief, all that has been said above may be formulated as follows. In all calculations of field theory one should use expressions for fields and their convolutions in the form of certain limits with respect to μ, and all other operations with these expressions should be performed under the sign $\lim_{\mu\to 0}$. In doing so, we can formulate the theory in such a way that at all stages of the calculations we deal only with ordinary functions and with proper operators of Hilbert space. This is connected with the fact that, before passing to the limit with respect to μ, we have only ordinary functions and states belonging to $\mathcal{H}$; then, in all expressions having physical meaning, we necessarily average these quantities over a certain space-time region. Thus we obtain a functional state of the type $|\Phi_g\rangle$. In this case it turns out that, under certain additional conditions on C_i and $\mathfrak{M}_i$, the limiting transition with respect to μ does not take the resulting functional states out of the Hilbert space $\mathcal{H}$. We have shown that, in the case of a renormalizable theory with a self-interacting scalar field, for the convergence with respect to μ of all expressions arising in the construction of the scattering matrix in perturbation theory, it is sufficient to require

$$\lim_{\mu\to 0} \sum_i C_i(\mu) \ln^\alpha \mathfrak{M}_i(\mu) = A_{0\alpha},$$
$$\alpha = 1, 2 \ldots \tag{17}$$
$$\lim_{\mu\to 0} \sum_i C_i(\mu)\mathfrak{M}_i(\mu) \ln^\alpha \mathfrak{M}_i(\mu) = A_{1\alpha},$$

In connection with the fact that we work only with normalizable state vectors, we nowhere encounter divergent expressions. In particular, with such an approach, not only do all matrix elements of the S-matrix (for finite bare charges in the masses) turn out to be finite, but, in contrast to the results of [1], the Hamiltonian is finite for a finite Lagrangian.

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[1] N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantum Fields*, Moscow, 1957. [2] R. Haag, *Dansk. Mat. Fys. Medd.*, **29**, No. 12 (1955).

Note: Figure translations are in progress. See original paper for figures.