D. L. BERMAN

1962

SovietRxiv

Abstract

Full Text

D. L. BERMAN

POLYNOMIAL OPERATIONS COMMUTATIVE WITH SHIFT

(Presented by Academician V. I. Smirnov on 6 January 1962)

1. Let us denote by $\widetilde{C}$ the space of all continuous 2π-periodic functions $f(x)$ with norm

$$\|f\| = \max_{0 \le x \le 2\pi} |f(x)|;$$

by Π_n the set of all trigonometric polynomials of order $\le n$; by M_n the set of all linear operations from $\widetilde{C}$ into $\widetilde{C}$ mapping $\widetilde{C}$ into Π_n; and by N_n the subset of M_n consisting of operators commutative with shift. Thus, if $U_n \in N_n$, then for any $f \in \widetilde{C}$ and for any $-\infty < t < \infty$ the equality $U(f_t) = [U_n(f)]_t$ holds, where $f_t(x) = f(x+t)$.

Let $U_n \in M_n$. Introduce the operator

$$\widetilde{U}_n(f, x) = \frac{1}{2\pi} \int_0^{2\pi} U_n(f_t, x - t)\, dt. \tag{1}$$

It is obvious that $U_n \in M_n$.

$\widetilde{U}_n$ has the following properties: 1) $\widetilde{U}_n \in N_n$; 2) if $U_n \in N_n$, then $\widetilde{U}_n = U_n$; this means that for any $f \in \widetilde{C}$ the equality $\widetilde{U}_n(f) = U_n(f)$ holds; 3) $\|\widetilde{U}_n\| \le \|U_n\|$. Properties 1) and 2) are obvious. Property 3) is proved in (1). We give examples.

Let $U_n(f) = L_n(f)$, where $L_n(f)$ is the Lagrange interpolation polynomial of order n, constructed for an arbitrary system of interpolation nodes $0 \le x_0 < x_1 < \dots < x_{2n} < 2\pi$; then $^{(2)}$ $\widetilde{U}_n(f) = S_n(f)$, where $S_n(f)$ is the partial sum of order n of the Fourier series of $f(x)$.

Let $U_n(f) = \lambda_n(f)$, where

$$\lambda_n(f, x) = \int_0^{2\pi} f(t) K_{n,p}(x, t)\, dt, \tag{2}$$

where $K_{n,p}(x, t)$ is a trigonometric polynomial of order n with respect to x and of order p with respect to t, $p \le n$; then

$$\widetilde{U}_n(f, x) = \int_0^{2\pi} f(z)\, dz \cdot \frac{1}{2\pi} \int_0^{2\pi} K_{n,p}(x - t, z - t)\, dt. \tag{3}$$

Let

$$U_n(f,x) = \frac{2\pi}{2n+1}\sum_{j=0}^{2n} f\left(\frac{2\pi j}{2n+1}\right) K_{n,p}\left(x, \frac{2\pi j}{2n+1}\right), \tag{4}$$

where $K_{n,p}(x,t)$ has the properties indicated earlier; then $\widetilde{U}_n$ is again found according to (3).

2. Let $\widetilde{U}_n^{\langle i\rangle} \in M_n$, $i = 1, 2$. Suppose that

$$U_n^{\langle 1\rangle}(f) = U_n^{\langle 2\rangle}(f), \qquad f \in \Pi_n. \tag{5}$$

It does not at all follow from equality (5) that $U_n^{\langle 1\rangle} = U_n^{\langle 2\rangle}$. However, the following theorem is true:

Theorem 1. If equality (5) holds on Π_n, then $\widetilde{U}_n^{\langle 1\rangle} = \widetilde{U}_n^{\langle 2\rangle}$. If $U_n^{\langle 2\rangle} \in N_n$, then $\widetilde{U}_n^{\langle 1\rangle} = U_n^{\langle 2\rangle}$.

Proof. By Lemma 2, from (3),

$$\widetilde{U}_n^{\langle i\rangle}(f) = \widetilde{U}_n^{\langle i\rangle}(S_n(f)), \quad i = 1, 2. \tag{6}$$

Since $S_n(f) \in \Pi_n$, by the hypothesis of the theorem,

$$U_n^{\langle 1\rangle}(S_n(f)) = U_n^{\langle 2\rangle}(S_n(f)). \tag{7}$$

The required assertion follows from (6) and (7). The second part of the theorem follows from the first part and from property 2) of the operator $\widetilde{U}_n$.

Special cases of Theorem 1 are known ([4]). Let $U_n \in M_n$ and $U_n(f) = \lambda_n(f)$, $f \in \Pi_n$, where $\lambda_n(f)$ is defined according to (2). Then, by Theorem 1, for any $f \in \widetilde{C}$,

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x-t)\,dt = \int_0^{2\pi} f(z)\,dz \int_0^{2\pi} K_{n,p}(x-t, z-t)\,dt.$$

Theorem 2. Suppose that (5) holds on Π_n, where $U_n^{\langle i\rangle} \in M_n$, $i = 1, 2$. Then

$$\|U_n^{\langle 1\rangle}\| \geq \|\widetilde{U}_n^{\langle 2\rangle}\|; \qquad \|U_n^{\langle 2\rangle}\| \geq \|\widetilde{U}_n^{\langle 1\rangle}\|. \tag{8}$$

Proof. By Theorem 1, $\widetilde{U}_n^{\langle 1\rangle} = \widetilde{U}_n^{\langle 2\rangle}$. Therefore $\|\widetilde{U}_n^{\langle 1\rangle}\| = \|\widetilde{U}_n^{\langle 2\rangle}\|$. By property 3) of the operator $\widetilde{U}_n$, we have $\|\widetilde{U}_n^{\langle i\rangle}\| \leq \|U_n^{\langle i\rangle}\|$. Consequently, $\|U_n^{\langle 1\rangle}\| \geq \|\widetilde{U}_n^{\langle 2\rangle}\|$. The second inequality in (8) is proved in the same way.

It is useful to compare this theorem with the results of ([3,5–7]).

In connection with Theorem 1 the question arises: in what case does the validity of equality (5) on Π_n imply its validity for any $f \in \widetilde{C}$? The answer to this question is given by Theorem 3.

Theorem 3. In order that the validity of equality (5) on Π_n imply its validity for any $f \in C$, it is necessary and sufficient that $U_n^{\langle 3\rangle} \in N_n$, where $U_n^{\langle 3\rangle} = U_n^{\langle 2\rangle} - U_n^{\langle 1\rangle}$.

Proof. Suppose equality (5) holds for any $f \in \widetilde{C}$. Then $U_n^{\langle 3\rangle}(f) = 0$. Therefore, for any $-\infty < t < \infty$,

$$U_n^{\langle 3\rangle}(f_t) = [U_n^{\langle 3\rangle}(f)]_t = 0.$$

Conversely, suppose $U_n^{\langle 3\rangle} \in N_n$. By property 2) of the operator $\widetilde{U}_n$, $U_n^{\langle 3\rangle} = \widetilde{U}_n^{\langle 3\rangle}$. By hypothesis, $U_n^{\langle 1\rangle}(f) = U_n^{\langle 2\rangle}(f)$, $f \in \Pi_n$; hence $U_n^{\langle 3\rangle}(f) = 0$, $f \in \Pi_n$. But in this case, by Theorem 1, $\widetilde{U}_n^{\langle 3\rangle}(f) = 0$ for any $f \in \widetilde{C}$; hence $U_n^{\langle 3\rangle} = 0$. Let $U_n \in M_n$. The question arises: when does there exist an operator from N_n that coincides on Π_n with U_n? The answer is given by Theorem 4.

Theorem 4. In order that, for a given $U_n \in M_n$, there exist an operator $U_n^{\langle 0\rangle} \in N_n$ coinciding on Π_n with the operator U_n, it is necessary and sufficient that the condition

$$\widetilde{U}_n(f) = U_n(f), \quad f \in \Pi_n \tag{9}$$

be satisfied.

If this condition is satisfied, then there exists a unique operator $\widetilde{U}_n$ satisfying the stated condition.

Proof. Suppose there exists an operator $U_n^{\langle 0\rangle} \in N_n$ coinciding on Π_n with the operator U_n. By Theorem 1, $\widetilde{U}_n^{\langle 0\rangle} = \widetilde{U}_n$. By property 2) of the operator $\widetilde{U}_n$, $\widetilde{U}_n^{\langle 0\rangle} = U_n^{\langle 0\rangle}$. It follows that $U_n^{\langle 0\rangle} = \widetilde{U}_n$.

Thus (9) has been established. From these arguments follows the uniqueness of the operator from N_n that coincides on Π_n with the given U_n. Sufficiency is obvious, since $\widetilde{U}_n \in N_n$.

Remark. With the help of Theorem 4 it is not difficult to see that if the kernel of the operator (2) satisfies the condition $K_{n,n}(x,t) \neq K_n(x-t)$, then for the operator (2) one cannot construct an operator from N_n satisfying condition (5).

3. Let, for the operator $U_n \in M_n$, there exist a convolution

$$\sigma_n(f,x) = \int_0^{2\pi} f(x+t)\Phi_n(t)\,dt,$$

where $\Phi_n(t)$ is a polynomial of order n, such that on Π_n the equality

$$U_n(f) = \sigma_n(f), \qquad f \in \Pi_n \tag{10}$$

holds. In this case, as is known (3), for any $f \in \widetilde{C}$ the equality

$$\widetilde{U}_n(f) = \sigma_n(f)$$

holds.

It turns out that requirement (10) is superfluous, for the following theorem is true.

Theorem 5. *Let $U_n \in M_n$; then for any $f \in \widetilde{C}$ the equality*

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x-t)\,dt = \int_0^{2\pi} f(x+t)\Phi_n(t)\,dt, \tag{11}$$

holds, where $\Phi_n(x) = \widetilde{U}_n[D_n(z)-x]$. ($D_n(z)$ is the Dirichlet kernel of order n.)

Proof. According to (3) we have

$$\widetilde{U}_n(f) = \widetilde{U}_n[S_n(f)]. \tag{12}$$

We compute the right-hand side

$$\widetilde{U}_n[S_n(f), x] = \frac{1}{2\pi}\int_0^{2\pi} U_n[(S_n(f))_t, x-t]\,dt. \tag{13}$$

Since

$$[S_n(f)]_t(z) = \int_0^{2\pi} f_t(t_1)D_n(z-t_1)\,dt_1,$$

it follows from (13) that

$$\widetilde{U}_n[S_n(f), x] = \frac{1}{2\pi}\int_0^{2\pi} dt \int_0^{2\pi} f_t(t_1)\,U_n[D_n(z-t_1), x-t]\,dt_1.$$

Therefore

$$\widetilde{U}_n[S_n(f), x] = \frac{1}{2\pi}\int_0^{2\pi} dt_1 \int_0^{2\pi} f(t+t_1)\widetilde{U}_n[D_n(z-t_1), x-t]\,dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} dt_1 \int_0^{2\pi} f(z_1)\widetilde{U}_n[D_n(z-t_1), x - z_1 + t]\, dz_1. \tag{14}$$

Since $\widetilde{U}_n \in N_n$, we have

$$\widetilde{U}_n[D_n(z-t_1), x - z_1 + t_1] = \widetilde{U}_n[D_n(z), x - z_1].$$

Thus, equality (14) takes the form

$$\tilde{U}_n[S_n(f),\, x] = \int_0^{2\pi} f(z_1)\Phi_n(z_1 - x)\, dz_1, \tag{15}$$

where

$$\Phi_n(x) = \tilde{U}_n[D_n(z),\, -x].$$

From equalities (12) and (15) the following theorem follows:

Theorem 6. For any $U_n \in M_n$ the inequality

$$\|U_n\| \geq \int_0^{2\pi} \left|\tilde{U}_n[D_n(z),\, x]\right| dx \tag{16}$$

holds.

Proof. From Theorem 5 it follows that $\|\tilde{U}_n\|$ is equal to the norm of the convolution on the right-hand side of (15), which is equal to the integral on the right-hand side of (15). To complete the proof it remains only to note that, by property 3) of the operator $\tilde{U}_n$, $\|\tilde{U}_n\| \leq \|U_n\|$.

For simplicity we have considered the case of the space $\tilde{C}$. The results of the note admit extension also to other functional spaces. They remain valid also in the case of functions defined on bicompact commutative groups.

Leningrad Institute of Soviet Trade
named after Fr. Engels

Received
4 I 1962

References

1. D. L. Berman, *Izv. vyssh. uchebn. zaved., Matematika*, **4** (17) (1960).
2. J. Marcinkiewicz, *Acta Szeged*, **8**, 127 (1937).
3. D. L. Berman, DAN, **85**, No. 1 (1952).
4. D. L. Berman, DAN, **92**, No. 4 (1953).
5. D. L. Berman, DAN, **88**, No. 1 (1953).
6. S. M. Lozinskii, DAN, **61**, No. 2 (1948).
7. S. M. Lozinskii, DAN, **89**, No. 5 (1953).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.