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Abstract

Full Text

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ON INADMISSIBLE VISCOSITY MATRICES FOR THE EQUATIONS OF ISOTHERMAL GAS MOTION

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MATHEMATICS

Of interest is the question (see ([1−3])) of the character of the dissipative terms, standing with a small multiplier ε on the right-hand side of the system of equations of isothermal gas motion

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0; \quad \mathbf{u} = (u, v); \quad \mathbf{f}(\mathbf{u}) = (p(v), -u), \quad p'(v) < 0; \quad p''(v) > 0, \tag{1}$$

for which, in the limit as $\varepsilon \to 0$, physically admissible discontinuous solutions are obtained.

I. M. Gel' fand in ([1]) put forward the hypothesis that as such terms one may take all possible "divergent viscosities"

$$\varepsilon \frac{\partial}{\partial x}\left[B(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}\right],$$

where $B(\mathbf{u})$ is a matrix, or even

$$\varepsilon \frac{\partial}{\partial x}\mathbf{b}\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}\right),$$

provided only that these terms ensure the "evolutionarity" of the arising system of equations of the second order:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left[B(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}\right]. \tag{2}$$

It should be noted, however, that at present an exact, non-"linearized" judgment about the presence or absence of this property in dissipative terms is difficult, even if the elements of the matrix B are constant (linear viscosities), because of the insufficiency of our knowledge of the Cauchy problem for systems (2), in particular for quasilinear parabolic systems (in this connection see the works

([4–6])). In any case, by analogy with systems with constant coefficients ([1]), for evolutionarity it appears necessary that the real parts of the eigenvalues of B be nonnegative.

In the present work we show that a certain class of linear viscosities satisfying this latter condition is inadmissible for gas dynamics.

We shall say that a certain viscosity matrix B is inadmissible for gas dynamics if, for system (2), there exists a sequence of solutions $\mathbf{u}_\varepsilon(x,t)$ converging as $\varepsilon \to 0$ to a rarefaction shock wave, i.e., to a discontinuous solution of system (1) for which at the discontinuity $v(x,t+0) > v(x,t-0)$.

Theorem. The matrix

$$B = \begin{pmatrix} a_1 & \alpha \\ 0 & b_1 \end{pmatrix},$$

where a_1, b_1 are arbitrary positive constants, and $\alpha \neq 0$ is any number, however close to zero, is inadmissible.

Proof. The solutions $\mathbf{u}_\varepsilon(x,t)$ will be chosen in the class of self-similar solutions —the so-called smeared discontinuities (see, for example, ([1])). The theorem will be proved if it is possible to choose such a rarefaction discontinuity for which the smeared discontinuities satisfy system (2). A smeared discontinuity is characterized by the parameters $u_-, v_-, u_+, v_+, \omega$, where u_-, v_- are the values of u, v to the left, and u_+, v_+ to the right of the straight line—

of the line of discontinuity, ω is the angular coefficient of the line of discontinuity with respect to the t-axis. As is known, these parameters are related by the relations

$$p(v_+) - p(v_-) = \omega(u_+ - u_-), \qquad -u_+ + u_- = \omega(v_+ - v_-). \tag{3}$$

Let us first consider the case $\alpha < 0$. Fix $\omega < 0$ so that the inequalities

$$a\omega + b\omega - \alpha > 0; \qquad \frac{1}{4ab}(a\omega + b\omega - \alpha)^2 - \omega^2 > 0, \tag{4}$$

hold, where $a = \dfrac{1}{a_1}$, $b = \dfrac{1}{b_1}$.

For this it is sufficient to take ω sufficiently close to zero. Next, the plus and minus signs on the parameters in (3) are arranged so that $v_- < v_+$ (then the discontinuity being smoothed will be a rarefaction shock). We shall show that a discontinuity with such parameters admits a smoothing $u(\xi) = u\left(\dfrac{x-\omega t}{\varepsilon}\right)$. The problem of finding $\mathbf{u}(\xi)$ has the form (see (1)):

$$B\frac{d\mathbf{u}}{d\xi} = \mathbf{f}(\mathbf{u}) - \omega\mathbf{u} - \mathbf{f}(\mathbf{u}_-) + \omega\mathbf{u}_-$$

or

$$\frac{d}{d\xi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & -\alpha \\ 0 & b \end{pmatrix}\begin{pmatrix} p(v) - \omega u + A \\ -u - \omega v + B \end{pmatrix}; \tag{5}$$

$$A = -p(v_-) + \omega u_-; \qquad B = u_- + \omega v_-; \tag{6}$$

$$u(-\infty) = u_-; \quad v(-\infty) = v_-; \quad u(+\infty) = u_+; \quad v(+\infty) = v_+.$$

To prove the solvability of this problem we shall use the results of G. Ya. Lyubarskii (7), for which purpose we transform problem (5), (6) to the analogous problem for a single second-order equation for $v(\xi)$. This problem has the form

$$-\frac{d^2v}{d\xi^2} - (a\omega + b\omega - \alpha)\frac{dv}{d\xi} - ab\varphi(v) = 0; \qquad v(-\infty) = v_-; \qquad v(+\infty) = v_+;$$

$$\varphi(v) = p(v) + \omega^2 v - p(v_-) - \omega^2 v_- \tag{7}$$

or

$$-\frac{d^2V}{d\xi^2} - (a\omega + b\omega - \alpha)\frac{dV}{d\xi} + F(V) = 0; \qquad V(-\infty) = 0; \qquad V(+\infty) = 1, \tag{8}$$

where

$$V(\xi) = \frac{v(\xi) - v_-}{v_+ - v_-}; \qquad F(V) = -\frac{ab}{v_+ - v_-}\varphi(v) > 0 \quad \text{for } 0 < V < 1;$$

$$F(0) = F(1) = 0.$$

The properties of the function $F(V)$ follow from the equalities (3) and the downward convexity of the function $p(v)$.

It is not difficult to verify that conditions (a), (b), and (c) from (7) are fulfilled. Let us dwell on condition (b).

In our case $a_{0,\max} = +\infty$; $a_{0,\min} = -\frac{1}{4}(a\omega + b\omega + \alpha)^2$. Condition (b) has the form:

$$a_{0,\min} < \frac{F(V_2) - F(V_1)}{V_2 - V_1} < a_{0,\max}, \qquad 0 \le V_1,\ V_2 \le 1.$$

In view of the convexity of $F(V)$ and the existence of $F'(V)$, this condition is equivalent to the condition

$$a_{0,\min} < F'(0), \quad F'(1) < a_{0,\max}.$$

But $F'(V) = -ab[p'(v) + \omega^2]$, so that condition (b) takes the form

$$p'(v_\pm) + \omega^2 < \frac{1}{4ab}(a\omega + b\omega - \alpha)^2.$$

The inequalities (4) and $p'(v) < 0$ ensure the fulfillment of this condition. Taking (4) into account, on the basis of Theorem I from ([7]) one may assert that there exists a solution $v(\xi)$ of problem (7), and with it also a solution $u(\xi), v(\xi)$ of problem (5), (6) (the inverse passage from (7) to (5), (6) is performed without difficulty).

The functions $u\left(\frac{x-\omega t}{\varepsilon}\right)$, $v\left(\frac{x-\omega t}{\varepsilon}\right)$ are a solution of system (2) and, as $\varepsilon \to 0$, tend to the original rarefaction discontinuity. This proves the theorem.

In the case $\alpha > 0$, the following changes are necessary in the reasoning. One should take $\omega > 0$, in (4) replace the first inequality by $a\omega + b\omega - \alpha < 0$, and take $v_+ < v_-$. Instead of (8) we shall have the problem

$$\frac{d^2V}{d\xi^2} + (a\omega + b\omega - \alpha)\frac{dV}{d\xi} + F(V) = 0; \qquad V(-\infty) = 0; \qquad V(+\infty) = 1;$$

$$F(0) = F(1) = 0; \qquad F(V) > 0 \text{ for } 0 < V < 1.$$

In this case $a_{0\,\min} = -\infty$; $a_{0\,\max} = \frac{1}{4}(a\omega + b\omega - \alpha)^2$. Condition (6) is again reduced to inequalities, which are fulfilled by virtue of the conditions on ω.

Remark 1. If $\alpha = 0$, then it is not difficult to show that the smearing problem is unsolvable for any rarefaction jump and, conversely, is solvable for any compression jump. The same is also true for real physical viscosity, i.e., when $\alpha = 0$; $b = 0$; $a = \zeta(v) > 0$.

Remark 2. We do not know whether the cause of the inadmissibility of the matrix B is the non-evolutionarity of system (2)*, or whether the hypothesis stated in ([1]) is false even for linear viscosities.

Remark 3. It would be interesting to obtain a result analogous to ([7]) in the case when a nonlinear coefficient stands at the first derivative. This would make it possible to investigate, by the method indicated here, symmetric viscosity matrices.

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* In the exact, and not in the "linearized," sense of this word.

Note: Figure translations are in progress. See original paper for figures.