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Abstract

Full Text

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ORTHOGONAL SYSTEMS OF RATIONAL FUNCTIONS ON THE CIRCLE WITH A PRESCRIBED SET OF POLES

In the present note we establish algebraic and, in part, limit properties of systems of rational functions orthonormal on the unit circle, all possible poles of which lie on a prescribed sequence of points outside the unit disk. In the limiting case, when all poles of the given system are identified with the infinitely remote point, the results of the present article reduce to the corresponding well-known assertions of the theory of polynomials orthogonal with a weight on the unit circle, developed in the works of G. Szegő ([1−3]).

1°. Let $\{\alpha_k\}_0^\infty$ ($|\alpha_k| < 1$) be an arbitrary sequence of complex numbers, among which numbers of finite or even infinite multiplicity may occur (not necessarily consecutively).

With the sequence $\{\alpha_k\}_0^\infty$ we associate two sequences of positive integers $\{\nu_k\}_0^\infty$ and $\{p_k\}_0^\infty$, where, for a given $k \geq 0$, ν_k denotes the multiplicity of occurrence of the number α_k in the group of numbers $\{\alpha_0, \alpha_1, \ldots, \alpha_k\}$, and p_k is defined by the conditions: $p_k = 1$, if $\alpha_k \neq 0$, $p_k = \nu_k$, if $\alpha_k = 0$ $(k = 0, 1, 2, \ldots)$. Finally, with the sequence of complex numbers $\{\alpha_k\}_0^\infty$ we put in correspondence the sequence of rational functions

$$\left\{z^{p_k - 1} / (1 - \overline{\alpha_k} z)^{\nu_k}\right\}_0^\infty . \tag{1}$$

Let us note that, for a given $k \geq 0$, the function $z^{p_k - 1}(1 - \overline{\alpha_k} z)^{-1}$ has only one pole of order ν_k at the point $z = 1/\overline{\alpha_k}$ ($|1/\overline{\alpha_k}| > 1$), if $\alpha_k \neq 0$; if $\alpha_k = 0$, then this pole lies at the point $z = \infty$ and has order $\nu_k - 1$.

We also observe that in the extreme case when $\alpha_k = 0$ $(k = 0, 1, 2, \ldots)$, obviously, $\nu_k = p_k = k + 1$, in view of which the zero sequence $\{0\}_0^\infty$ will be put in correspondence with the sequence of polynomials $\{z^k\}_0^\infty$.

By elementary considerations it is easy to verify that the system $\{\varphi_k(z)\}_0^\infty$ of Malmquist rational functions ([4,5])

$$\varphi_0(z) = \frac{(1-|\alpha_0|^2)^{1/2}}{1-\overline{\alpha_0}z}, \quad \varphi_n(z) = \frac{(1-|\alpha_n|^2)^{1/2}}{1-\overline{\alpha_n}z} \prod_{k=0}^{n-1} \frac{\alpha_k - z}{1-\overline{\alpha_k}z} \frac{|\alpha_k|}{\alpha_k}^{*} \quad (k=1,2,...), \tag{2}$$

orthogonal on the unit circle in the sense of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(z)\overline{\varphi_m(z)}\,dx = S_{nm} = \begin{cases} 0, & m \neq n, \\ 1, & m = n \end{cases} \quad (n,m=0,1,2,\dots;\ z=e^{ix}), \tag{3}$$

is the result of orthogonalizing the ordered sequence of functions (1) on the unit circle $z = e^{ix}$ $(-\pi \le x \le \pi)$.

2°. Let $\alpha(x)$ be an arbitrary bounded nondecreasing function on the interval $[-\pi, \pi]$ with an infinite set of points of increase. Orthogonalizing the ordered sequence of rational functions (2) on the circle $z = e^{ix}$ $(-\pi \le x \le \pi)$ in the presence of the weight $2\pi^{-1}d\alpha(x)$, we obtain a system of rational functions $\{\Phi_k(z)\}_0^\infty$ satisfying conditions which determine the functions of our system uniquely.

a) $\Phi_n(x)$ is "of full order n" with respect to the first $n+1$ Malm–

* Here and in what follows it is assumed that, for $\alpha_k = 0$, $|\alpha_k|/\alpha_k = \overline{\alpha_k}/\alpha_k = -1$.

orthonormalization

$$\Phi_n(z) = \sum_{k=0}^{n} c_{k,n}\varphi_k(z), \quad c_{n,n} > 0; \qquad \text{b)}\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_n(z)\overline{\Phi_m(z)}\,d\alpha(x) = \delta_{nm},$$

$$z = e^{ix} \qquad (n,m = 0,1,2,...).$$

Next, denote by $\{\Phi_k^{(n)}(z)\}_0^n$ the result of orthonormalizing the finite system of functions

$$\varphi_k^{(n)}(z) = -\frac{B_{n+1}(z)}{z}\overline{\varphi_{n-k}\left(\frac{1}{\bar{z}}\right)} \qquad (k=0,1,2,\dots,n) \tag{4}$$

with the same weight $(2\pi)^{-1}d\alpha(x)$, where again we assume that in the representations

$$\Phi_k^{(n)}(z) = \sum_{p=0}^{k} c_{p,k}^{(n)} \varphi_p^{(n)}(z) \qquad (k = 0, 1, \dots, n) \tag{5}$$

the leading coefficients $c_{k,k}^{(n)}$ $(k = 0, 1, \dots, n)$ are positive. Introduce the following notation for the coefficients of the leading terms and for the constant terms in formulas (4) and (5): $c_{n,n} = k_n$, $c_{0,n} = l_n$, $c_{k,k}^{(n)} = k_k^{(n)}$, $c_{0,k}^{(n)} = l_k^{(n)}$, and consider the kernels of distribution $(2\pi)^{-1} d\alpha(x)$

$$S_n(\zeta, z) = \sum_{k=0}^{n} \overline{\Phi_k(\zeta)}\, \Phi_k(z) \qquad (n = 0, 1, 2, \dots) \tag{6}$$

and the partial Blaschke products

$$B_{n+1}(z) = \prod_{k=0}^{n} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k}. \tag{7}$$

Theorem 1. *For every $n \geq 0$ the following formulas hold:*

a)

$$S_n(\alpha_n, z) = -\frac{|\alpha_n|}{\alpha_n} \frac{k_n}{(1 - |\alpha_n|^2)^{1/2}} \frac{B_{n+1}(z)}{z} \overline{\Phi_n\left(\frac{1}{\bar{z}}\right)}, \qquad S_n(\alpha_n, \alpha_n) = \frac{k_n^2}{1 - |\alpha_n|^2}.$$

b)

$$S_n(\alpha_0, z) = -\frac{|\alpha_0|}{\alpha_0} \frac{k_n^{(n)}}{(1 - |\alpha_0|^2)^{1/2}} \frac{B_{n+1}(z)}{z} \overline{\Phi_n^{(n)}\left(\frac{1}{\bar{z}}\right)}, \qquad S_n(\alpha_0, \alpha_0) = \frac{(k_n^{(n)})^2}{1 - |\alpha_0|^2}.$$

c) *The coefficients $\{l_p^{(n)}\}_0^n$ and k_n, as well as $\{l_p\}_0^n$ and $k_n^{(n)}$, are connected by the relations*

$$\sum_{p=0}^{n} |l_p^{(n)}|^2 = k_n^2, \qquad \sum_{p=0}^{n} |l_p|^2 = (k_n^{(n)})^2.$$

Taking into account that the linear manifolds $\{\Phi_k(z)\}_0^n$ and $\{\Phi_k^{(n)}(z)\}_0^n$ coincide, one proves

Theorem 2. *For any z and ζ, when $n \geq 0$, the kernel $S_n(\zeta, z)$ of the distribution $(2\pi)^{-1} d\alpha(x)$ satisfies the functional equation*

$$S_n(\zeta, z) = \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{\bar{\zeta} z} \, S_n\left(\frac{1}{\bar{z}}, \frac{1}{\bar{\zeta}}\right). \tag{8}$$

3°. We denote by $M\{\alpha_k\}_0^n$ all possible linear combinations of the functions of the system

$$\left\{ \frac{z^{p_k - 1}}{(1 - \alpha_k z)^{\nu_k}} \right\}_0^n.$$

As for ordinary polynomials, the following property of the kernel $S_n(\zeta, z)$ holds.

Let, for arbitrary values of the parameter $\zeta \neq 1/\bar{\alpha}_n$ $(k = 0, 1, \dots, n)$, $\rho(\zeta, z) \in M\{\alpha_k\}_0^n$. In order that, for every function $g(z) \in M\{\alpha_k\}_0^n$, the identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\zeta, z) \overline{g(z)} \, d\alpha(x) = \overline{g(\zeta)}, \qquad z = e^{ix},$$

hold, it is necessary and sufficient that $\rho(\zeta, z) = S_n(\zeta, z)$.

On the basis of this property, one establishes

Theorem 3. For arbitrary z and ζ, $n \geq 0$, for the kernel $S_n(\zeta, z)$ of the distribution $(2\pi)^{-1} d\alpha(x)$ Christoffel' s formula is valid

$$S_n(\zeta, z) = \frac{(1 - \overline{\alpha}_{n+1}\zeta)(1 - \overline{\alpha}_{n+1} z)}{1 - |\alpha_{n+1}|^2} \, \frac{\Phi_{n+1}^*(\zeta)\Phi_{n+1}^*(z) - \overline{\Phi}_{n+1}(\zeta)\Phi_{n+1}(z)}{1 - \overline{\zeta} z}, \tag{9}$$

where

$$\Phi_{n+1}^*(z) = \frac{B_{n+2}(z)}{z} \, \overline{\Phi}_{n+1}\left(\frac{1}{z}\right), \qquad B_{n+2}(z) = \prod_{k=0}^{n+1} \frac{\alpha_k - z}{1 - \overline{\alpha}_k z} \, \frac{|\alpha_k|}{\alpha_k}.$$

Using formula (9) and Theorem 1, one can establish the following recurrence relation, which is satisfied by the functions of the system $\{\Phi_k(z)\}_0^\infty$:

$$\begin{aligned} &-\frac{|\alpha_{n+1}|}{\alpha_{n+1}} \, \frac{k_{n+1}}{(1 - |\alpha_{n+1}|^2)^{1/2}} \, \Phi_{n+1}(z) \\ &= \frac{B_{n+2}(z)}{z} \left\{ \Phi_{n+1}(\alpha_{n+1}) \, \overline{\Phi}_{n+1}\left(\frac{1}{z}\right) + \sum_{k=0}^{n} \Phi_k(\alpha_{n+1}) \, \overline{\Phi}_k\left(\frac{1}{z}\right) \right\}. \end{aligned} \tag{10}$$

For $\alpha_{n+1} \neq 0$ this relation, after transformations, takes the form

$$\frac{|\alpha_{n+1}|}{\alpha_{n+1}} \, \frac{k_{n+1}}{(1 - |\alpha_{n+1}|^2)^{1/2}} \, \Phi_{n+1}(z) = \Phi_{n+1}(\alpha_{n+1}) \, \frac{B_{n+2}(z)}{z} \, \overline{\Phi}_{n+1}\left(\frac{1}{z}\right) +$$

$$+\frac{|\alpha_{n+1}|}{\alpha_{n+1}}\,\frac{B_{n+1}(\alpha_{n+1})}{\alpha_{n+1}}\,\frac{\alpha_{n+1}-z}{1-\overline{\alpha}_{n+1}z}\sum_{k=0}^{n}\overline{\Phi}_k\left(\frac{1}{\alpha_{n+1}}\right)\Phi_k(z). \tag{10'}$$

4°. The limiting properties of the system $\{\Phi_k(z)\}_0^\infty$, in particular of the kernels $S_n(\zeta,z)$, depend essentially on whether the quantities*

$$A=\int_{-\pi}^{\pi}\log\alpha'(x)\,dx,\qquad B=\sum_{k=0}^{\infty}(1-|\alpha_k|)$$

are finite or not.

The following proposition is a generalization of a well-known theorem of G. Szegő ([1–3]).

Theorem 4. Let the sequence of numbers $\{\alpha_k\}_0^\infty$ $(|\alpha_k|<1)$ be such that

$$\sum_{k=0}^{\infty}(1-|\alpha_k|)=+\infty. \tag{11}$$

Then: a) If

$$\int_{-\pi}^{\pi}\log\alpha'(x)\,dx>-\infty, \tag{12}$$

then

$$S(\zeta,z)=\lim_{n\to\infty}S_n(\zeta,z)=\frac{1}{(1-\bar\zeta z)D(\bar\zeta)D(z)}\qquad(|z|<1,\ |\zeta|<1), \tag{13}$$

where

$$D(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\alpha'(t)\,\frac{e^{it}+z}{e^{it}-z}\,dt\right\}.$$

b) Under condition (22), if

$$\mu_n(\zeta)=\min_{\{Q_n\}}\frac{1}{2\pi}\int_{-\pi}^{\pi}|Q_n(z)|^2\,d\alpha(x),\qquad z=e^{ix}, \tag{14}$$

* We note that in the metrics $L_p(d\alpha(x),-\pi,\pi)$ $(p>0)$ questions of approximation by rational functions of the system

$$\left\{\frac{1}{1-\overline{\alpha}_k z}\right\}_0^\infty$$

in the cases $A > -\infty,\ B < +\infty;\ A > -\infty,\ B = +\infty$ and $A = -\infty,\ B = +\infty$ were investigated by G. Ts. Tumarkin ([6]).

where

$$Q_n(z) = \sum_{k=0}^{n} u_k \varphi_k(z), \quad Q_n(\zeta) = 1 \quad (|\zeta| < 1),$$

then

$$\lim_{n\to\infty} \mu_n(\zeta) = \frac{1}{S(\zeta,\zeta)} = (1 - |\zeta|^2)|D(\zeta)|^2. \tag{15}$$

c) If

$$\int_{-\pi}^{\pi} \log \alpha'(x)\, dx = +\infty, \tag{16}$$

then

$$\lim_{n\to\infty} \mu_n(\zeta) = \frac{1}{S(\zeta,\zeta)} = 0. \tag{17}$$

Thus, under condition (11), the asymptotic properties of the kernels $S_n(\zeta, z)$ and the minima $\mu_n(\zeta)$ of the generalized Toeplitz forms do not differ from the case $\alpha_k = 0$ $(k = 0, 1, 2, ...)$, considered by G. Szegő. In the case, however, when

$$\sum_{k=0}^{\infty} (1 - |\alpha_k|) < +\infty,$$

the behavior of the kernels $S_n(\zeta, z)$ (or of the minima $\mu_n(\zeta)$) changes substantially. For example, assuming that the numbers $\{\gamma_k\}_0^p$ $(|\gamma_k| < 1)$ are arbitrary, consider the functions

$$\omega_p(z) = A \prod_{k=0}^{p} \frac{z - \gamma_k}{1 - \overline{\alpha}_k z}, \quad D_p(z) = A \prod_{k=0}^{p} \frac{1 - \overline{\alpha}_k z}{1 - \gamma_k z}$$

$(A > 0)$ and the distribution $d\alpha(x) = w_p(x)\, dx$, where

$$w_p(x) = |D_p(e^{ix})|^2.$$

Then the following is valid.

Theorem 5. a) The system of orthogonal rational functions $\{\widetilde{\Phi}_n(z)\}_0^\infty$, associated with the distribution $d\alpha(x) = w_p(x)\,dx$, admits the representation

$$\widetilde{\Phi}_n(z) = e^{i\gamma_n}\omega_p(z)\frac{(1-|\alpha_n|^2)^{1/2}}{1-\overline{\alpha}_n z}\prod_{k=p+1}^{n-1}\frac{\alpha_k - z}{1-\overline{\alpha}_k z}\frac{|\alpha_k|}{\alpha_k} \quad (n = p+1, p+2, \ldots), \tag{18}$$

where γ_n are constants. b) For any z and ζ not lying on the unit circle and at the points of the sequence $\{1/\overline{\alpha}_k\}_0^\infty$, the formulas

$$S(\zeta, z) = \lim_{n\to\infty} S_n(\zeta, z) = \frac{1}{(1-\overline{\zeta})\overline{D}_p(\zeta)D_p(z)} - \frac{\overline{B}(\zeta)B(z)}{(1-\overline{\zeta}z)\overline{D}_p\left(\frac{1}{\zeta}\right)D_p\left(\frac{1}{z}\right)}, \tag{19}$$

$$\lim_{n\to\infty}\mu_n(\zeta) = (1-|\zeta|^2)|D_p(\zeta)|^2\left\{1-\left|B(\zeta)\frac{D_p(\zeta)}{\overline{D}_p(1/\zeta)}\right|^2\right\}^{-1}, \tag{20}$$

hold, where

$$B(z) = \prod_{k=0}^{\infty}\frac{\alpha_k - z}{1-\overline{\alpha}_k z}\frac{|\alpha_k|}{\alpha_k}$$

is a convergent Blaschke product.

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Note: Figure translations are in progress. See original paper for figures.