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1962

SovietRxiv

Abstract

Full Text

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On the Semisimplicity of the Semigroup of Endomorphisms of Ordered Sets

(Presented by Academician A. I. Maltsev on 8 VIII 1961)

Let Ω be an ordered set (an equivalent term is a partially ordered set). According to the general definition of an endomorphism of a set in which certain relations are specified, a transformation S of the set Ω is called an endomorphism of Ω if from $\alpha \leq \beta$ ($\alpha, \beta \in \Omega$) it always follows that $S\alpha \leq S\beta$. It is known that the set of all endomorphisms of Ω is a semigroup with respect to the usual multiplication of transformations ($S_1 S_2(\alpha) = S_1(S_2(\alpha))$) ([3]). In what follows, by Ω we shall denote an ordered set, and by Σ the semigroup of all its endomorphisms.

It is of interest to establish a connection between the properties of the set Ω and those of the semigroup Σ. Some results in this direction have already been obtained. Thus, L. M. Gluskin proved that the order relation in Ω is completely determined by the semigroup Σ, apart from one trivial case ([1]). The present note is also devoted to this direction.

Let $\mathfrak{A}$ be an arbitrary semigroup, and I its ideal (two-sided). We say that $a, b \in \mathfrak{A}$ are comparable modulo the ideal I if $a = b$ or $a, b \in I$. Comparison modulo the ideal I is a two-sided stable equivalence; it is usually called an ideal equivalence. A semigroup $\mathfrak{A}$ is called semisimple if all its two-sided stable equivalences distinct from equality are ideal ([2]). It is known that finding two-sided stable equivalences in a semigroup is equivalent to finding its homomorphisms up to isomorphism.

In the article a characterization of a finite linearly ordered set is indicated with the aid of properties of the semigroup Σ.

We shall call the elements of the semigroup Σ transformations. The rank of a transformation $S \in \Sigma$, as usual, will mean the cardinality of the set $S\Omega$, and will be denoted by rS.

The simplest notions of semigroup theory may be found in the book ([3]).

§ 1. In this section we shall assume Ω to be a finite linearly ordered set consisting of n elements. We shall describe all two-sided stable equivalences of the semigroup Σ, using the approach to the description of homomorphisms from ([4]).

Let $k \leq n$. Denote by I_k the set of all transformations in Σ whose ranks do not exceed k.

Theorem 1. *The subsets I_k, and only they, are ideals of the semigroup Σ.*

In the proof of the main theorem, Lemmas 1, 2, and 3 are used.

Lemma 1. *Under any two-sided stable equivalence distinct from equality, all transformations of unit rank are equivalent to one another.*

Lemma 2. Under any two-sided stable equivalence, substitutions equivalent to substitutions of unit rank form an ideal of the semigroup Σ.

From Theorem 1 and Lemma 2 there follows

Corollary. If, under some two-sided stable equivalence, there are two equivalent substitutions A, B such that $rA = 1,\ rB = k > 1$, then all substitutions of rank not exceeding k will be equivalent to one another.

Lemma 3. If, under some two-sided stable equivalence, a substitution A is not equivalent to substitutions of unit rank and a substitution B is equivalent to A, then $rA = rB$.

Theorem 2. If Ω is a finite linearly ordered set, then the semigroup Σ of all endomorphisms of Ω is semisimple.

§ 2. In the semigroup Σ we define a relation $\mathfrak{R}$. $A \sim B$ ($\mathfrak{R}$), if one of the following conditions is satisfied: 1) A, B are substitutions of finite rank; 2) A, B are of infinite rank, and if Ω_0 is the set of those elements of Ω which the substitutions A, B transform differently, then the sets $A\Omega_0,\ B\Omega_0$ are finite. The relation $\mathfrak{R}$ is a two-sided stable equivalence on Σ ([4]).

Let Ω be an infinite linearly ordered set. We shall write the elements of the semigroup Σ in the form of substitutions. An element of Ω to which a substitution sends only itself will sometimes be omitted in writing the substitution.

We shall prove that the equivalence $\mathfrak{R}$ is not ideal. For this it is enough to prove that there exist at least two $\mathfrak{R}$-classes containing more than one element. One such class is formed by all substitutions of finite rank.

Let $m_1 < m_2$ be elements of Ω, chosen so that the number of elements $\alpha < m_1$ or the number of elements $\beta > m_2$ is infinite. It is easy to show that such elements m_1, m_2 can be found in any infinite linearly ordered set.

The substitutions

$$A = \left(\frac{\overbrace{m_1 \dots m_2}}{m_1} \right), \qquad B = \left(\frac{\overbrace{m_1 \dots m_2}}{m_2} \right),$$

where the dots are understood to denote all $\gamma \in \Omega$ such that $m_1 < \gamma < m_2$, have infinite ranks and, consequently, belong to a second $\mathfrak{R}$-class containing more than one element.

Thus, it has been proved:

Theorem 3. If Ω is an infinite linearly ordered set, then the semigroup Σ of all endomorphisms of Ω is not semisimple.

Let the ordering in Ω not be linear. To prove that the semigroup Σ is not semisimple, we shall need one auxiliary theorem.

Let

$$S = \begin{pmatrix} \Omega_1 & \Omega_2 & \dots & \Omega_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix} \in \Sigma,$$

where Ω_i is the set of elements of Ω transformed by the substitution S into a_i, all $a_1, a_2, \dots, a_k$ are distinct, and $\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_k$. We shall say that the substitution S effects a partition of Ω into the subsets $\Omega_1, \Omega_2, \dots, \Omega_k$.

Theorem 4. In order that the ordering in the set Ω be non-linear, it is necessary and sufficient that in the semigroup Σ there exist two distinct substitutions A, B satisfying the conditions: 1) A, B effect the same partition of Ω into subsets; 2) $A\Omega = B\Omega$.

Define in the semigroup Σ the relation L_k for any natural number k. Let $A \sim B$ (L_k) if one of the following conditions is satisfied: 1) $A = B$; 2) the ranks of the substitutions A, B are less than k; 3) $rA = rB = k$, the substitutions A, B effect the same partition Ω into subsets, and $A\Omega = B\Omega$. It is easy to show that the relation L_k is a two-sided stable equivalence.

From Theorem 4 we obtain that, for some k, the two-sided stable equivalence L_k is not ideal. Hence it follows:

Theorem 5. *If the ordering on Ω is not linear, then the semigroup Σ is not semisimple.*

Combining the results given above, we obtain:

Theorem 6. *In order that the ordered set Ω be finite linearly ordered, it is necessary and sufficient that the semigroup of all endomorphisms of Ω be semisimple.*

Received
1 VIII 1961

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Note: Figure translations are in progress. See original paper for figures.