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Abstract

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RELATIVELY ELEMENTARY SUBSPACES IN COMPACT LIE ALGEBRAS

(Presented by Academician A. I. Mal' cev, 1 II 1962)

In the work ([1]) A. I. Mal' cev introduced the following definition. A subset T of the elements of an algebra R is called **relatively elementary** if there exist elements $e_1, \dots, e_m$ of the algebra R and a formula $\mathfrak{A}(x, y_1, \dots, y_m)$ of the narrow predicate calculus with equality, containing no predicate symbols other than the equality symbol, containing no functional symbols other than the symbols for addition and multiplication, and containing no free variables other than $y_1, \dots, y_m, x$, such that $\mathfrak{A}(x, e_1, \dots, e_m)$ is true for those and only those $x \in R$ that are contained in T. A. I. Mal' cev ([1]) posed the question of the relative elementarity of subalgebras in simple compact Lie algebras; a positive answer to this question is given in the present note.

Theorem 1. Let R be a simple real compact Lie algebra, and let R_1 be a subspace of the space R. Then R_1 is relatively elementary.

Theorem 2. Let R be a simple complex Lie algebra, and let R_1 be a subspace of the space R. Then R_1 is relatively elementary.

Theorem 3. A real compact Lie algebra is simple if and only if all its subspaces are relatively elementary.

Theorem 4. A complex semisimple Lie algebra is simple if and only if all its subspaces are relatively elementary.

The proof of Theorems 3 and 4 is based on the following lemma.

Lemma 1. Let an algebra R over an infinite field K decompose into the direct sum of its two-sided ideals $R_1, \dots, R_k$. Let S be a subspace of the space R, relatively elementary in the algebra R; let $S_1 = S \cap R_1, \dots, S_k = S \cap R_k$. Then

$$S = S_1 + \dots + S_k.$$

The vector space R over the field K may be regarded as an algebra with zero multiplication. The relatively elementary subsets of this algebra are called **relatively elementary subsets of the space** R.

Lemma 2. Let the characteristic of the infinite field K be different from two, let R be a vector space over the field K, and let S be a subspace of the space R relatively elementary in R. Then either S is R, or S is the zero subspace.

Lemma 3. Let an algebra R over an infinite field K, whose characteristic is different from two, decompose into the direct sum of two-sided ideals R_0 with zero multiplication and R_1. Let S be a relatively elementary subspace of the algebra R. Then $S \cap R_0 = \{0\}$ or $S \cap R_0 = R_0$.

Theorem 5. Let R be a real compact Lie algebra; R_0 the center of the algebra R; $R_1, \dots, R_k$ the simple noncommutative ideals of the algebra R;

$$R = R_0 + R_1 + \dots + R_k.$$

Let S be a subspace of R, and let $S \cap R_i = S_i$ $(i = 0, 1, \dots, k)$. The subspace S is relatively elementary in R if and only if: 1) $S_0 = R_0$ or $S_0 = \{0\}$, and 2)

$$S = S_0 + S_1 + \dots + S_k.$$

The proof uses Theorem 1, Lemmas 1, 3, and the following lemma.

Lemma 4. *The ideals $R_0, R_1, \dots, R_k$ are relatively elementary in R (notation as in Theorem 5).*

Corollary. In the notation of Theorem 5, if $R = R_0 \dot{+} R_1$, then every semisimple subalgebra of the algebra R is relatively elementary.

Theorem 6. *Let R be a complex semisimple Lie algebra; let $R_1, \dots, R_k$ be the simple ideals of the algebra R; let $R = R_1 \dot{+} \dots \dot{+} R_k$; let S be a subspace of the space R; and let $S \cap R_i = S_i$ $(i = 1, \dots, k)$. The subspace S is relatively elementary in R if and only if*

$$S = S_1 \dot{+} \dots \dot{+} S_k.$$

The proof uses Theorem 2, Lemma 1, and the following lemma.

Lemma 5. *The ideals $R_1, \dots, R_k$ are relatively elementary in R (notation as in Theorem 6).*

In conclusion, I express my gratitude to Academician A. I. Mal' cev for posing the problem.

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REFERENCES

1. A. I. **Mal' cev**, Some problems of mathematics and mechanics, 1961, pp. 112–134.

Note: Figure translations are in progress. See original paper for figures.