ON BENDING SURFACES OF POSITIVE CURVATURE UNDER CERTAIN BOUNDARY CONDITIONS

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1962

SovietRxiv

Abstract

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MATHEMATICS

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ON BENDING SURFACES OF POSITIVE CURVATURE UNDER CERTAIN BOUNDARY CONDITIONS

(Presented by Academician I. N. Vekua, 16 VIII 1961)

Let an unclosed surface with coefficients b_{ij} of the second fundamental form be given in Euclidean space. In investigations of the non-bendability of surfaces with boundary under certain boundary conditions, one can use the system of Gauss and Codazzi equations, regarding in it the coefficients of the second form of the surface as unknown functions, while considering the coefficients of the metric form as prescribed. Any theorem on the absence of a solution of some boundary-value problem in a neighborhood of a solution b_{ij} would imply the non-bendability of the given piece of surface under the prescribed boundary condition.

1. Under an isothermally conjugate parametrization

$$II = b_0(u,v)(du^2 + dv^2), \quad b_0 > 0,$$

of the initial surface of positive Gaussian curvature K with $r(u,v) \in D_{3,p}(D+\Gamma)$, $p > 2$, where the domain D is canonical (1), Γ is its boundary, and with edge

$$L = \sum_{i=0}^{m} L_i \in C^1_{\mu,\nu_1\ldots\nu_k} \quad (0 < \mu < 1,\ \nu_i > 0),$$

the Gauss and Codazzi equations are represented in the following form:

$$\partial_{\bar z} W(z) + A(z)W(z) + B(z)\overline{W(z)} = C(z)\left(-b_0(z) + \sqrt{b_0^2(z) + W(z)\overline{W(z)}}\right) + i\partial_z\left(-b_0(z) + \sqrt{b_0^2(z) + W(z)\overline{W(z)}} \tag{1}$$

where $z = u + iv$;

$$W = b_{12} + \frac{i}{2}(b_{11} - b_{22}),$$

and the coefficients A, B, C are expressed in a known way through the Christoffel symbols and belong to the class

$$C_{\frac{p-2}{p}}(D+\Gamma).$$

Many geometric contour relations can be expressed in the form of the following boundary condition:

$$\operatorname{Re}\{\overline{\lambda(t)}\,W(t)\} + \Phi(W;t) = \gamma(t), \quad t \in \Gamma, \tag{2}$$

where $\lambda(t)$, $\gamma(t)$ are prescribed and belong to the class $C_\sigma(\Gamma)$, $\sigma \geq \frac{p-2}{p}$, $p > 2$; the prescribed function $\Phi(W;t)$ satisfies the conditions:

I.

$$C_\alpha(\Phi(W;t)) \leq \rho\,\nu(\rho) \quad \text{for } C_\alpha(W) \leq \rho, \quad \text{where } \lim_{\rho\to 0} \nu(\rho) = 0, \quad \alpha = \frac{p-2}{p}.$$

II.

$$C_\alpha(\Phi(W_2;t) - \Phi(W_1;t)) \leq \mu(\rho)\,C_\alpha(W_2 - W_1) \quad \text{for } C_\alpha(W_i) \leq \rho,$$

where

$$\lim_{\rho\to 0} \mu(\rho) = 0.$$

Consider the boundary-value problem (1), (2) with sought function $W(z)$. Let us call the index of the problem the number

$$n = \frac{1}{2\pi}\Delta_\Gamma \arg \lambda(t).$$

Theorem 1. *If $n < 0$, then the boundary-value problem (1), (2) has in a neighborhood of zero no more than one solution belonging to the class $D_{1,p}(\dot{D}+\Gamma)$, $p > 2$, provided that $C_\sigma(\gamma;\Gamma) < \varepsilon$, where ε is a sufficiently small positive number. If $n > m-1$, then the boundary-value problem (1), (2) is always solvable under*

for any choice of the function $\gamma(t)$ under the condition that $C_\sigma(\gamma;\Gamma) < \varepsilon$; the solution depends on $2n+1-m$ arbitrary constants.

The proof of the theorem is carried out by the method of paper ([2]).

§ 2. Using the results of § 1, we can, for example, investigate the case of bending a piece of a surface of positive curvature with preservation, along the edge, of the principal directions or of the mean curvature.

Theorem 2. Let the edge L of an $(m+1)$-connected piece of a surface of positive curvature of class $D_{3,p}(D+\Gamma)$, $p>2$, contain no umbilic points and make, with one of the families of principal directions, an angle

$$\vartheta(s) \in C_{\frac{p-2}{p}}.$$

If

$$\varkappa = \frac{1}{2\pi} \Delta_L \vartheta < 1-m,$$

then a bending of the surface preserving the principal directions or the mean curvature of the surface along the edge reduces to a motion. If $2\varkappa > 1-m$, then there exists at most—and for $m=0$, exactly—a $(4\varkappa - 3 + 3m)$-parameter family of bendings of the surface preserving the principal directions or the mean curvature of the surface along the edge.

The case $\vartheta = 0,\ \vartheta = \pi/4$ was considered by K. M. Belov in ([3]). Let us outline the proof.

It can be shown that the geodesic torsion τ_g and the normal curvature k_n along the edge are related to the angle $\vartheta(s)$ by the relation

$$k_n^2 \sin 2\vartheta + 2\tau_g k_n \cos 2\vartheta - \tau_g^2 \sin 2\vartheta = K \sin 2\vartheta.$$

This relation must hold identically on any surface. But if we require that the angle $\vartheta(s)$ be an invariant of bending, we obtain the boundary condition:

$$(k_n \cos 2\vartheta - \tau_g \sin 2\vartheta)\Delta\tau_g + (k_n \sin 2\vartheta + \tau_g \cos 2\vartheta)\Delta k_n +$$

$$+(\Delta k_n^2 - \Delta\tau_g^2)\sin\vartheta\cos\vartheta + \Delta\tau_g \Delta k_n \cos 2\vartheta = 0, \tag{3}$$

where $\Delta\tau_g$ and Δk_n are expressed in terms of $W(z)$ as follows:

$$\Delta\tau_g = \mathrm{Re}\{\overline{\lambda_1(t)}\, W(t)\} + \psi_1(t)\left(-b_0(t) + \sqrt{b_0^2(t) + W(t)\overline{W(t)}}\right),$$

$$\Delta k_n = \mathrm{Re}\{\overline{\lambda_2(t)}\, W(t)\} + \psi_2(t)\left(-b_0(t) + \sqrt{b_0^2(t) + W(t)\overline{W(t)}}\right). \tag{4}$$

Here $\lambda_i(t)$ and $\psi_i(t)$ are known functions.

Substituting (4) into (3), we obtain for W a boundary condition of the form (2), whose index is equal to $n = \varkappa + 2(m-1)$, and the nonlinear terms form a function $\Phi(W,t)$ satisfying the requirements listed above. Hence, by Theorem 1, the validity of the first part of Theorem 2 follows. The proof of the second part is carried out in the same way as the preceding one; however, to derive the boundary condition one should start from the relation $III - 2HII + KI = 0$, requiring that H be an invariant of bending.

As an example, we note that in the case of a simply connected piece of a surface of positive curvature $\varkappa = 0$, if $-\pi < \vartheta < \pi$. This, in particular, is fulfilled for surfaces of revolution when the edge encircles the axis of rotation; in other cases $\varkappa = 1$.

Remark. Theorem 2 is formulated analogously for infinitesimal bendings. It makes it possible, for concrete examples, to establish the necessity of satisfying condition (7.76) indicated by I. N. Vekua in Theorem 5.11 ([1]).

§ 3. Consider on the surface some direction (l), given along the edge and making with it an angle $\varphi(s) \in C_{\frac{p-2}{p}},\ p > 2$.

Theorem 3. Let the angle $\varphi(s)$ be chosen so that one of the relations holds:

$$\sin^2\varphi(s) < \frac{k_n(s)}{2H(s)}; \qquad \sin^2\varphi(s) = \frac{k_n(s)}{2H(s)}; \qquad \sin^2\varphi(s) > \frac{k_n(s)}{2H(s)}. \tag{5}$$

If, under a bending of a simply connected piece of a surface of positive curvature of class $D_{3,p}$, $p > 2$, the normal curvature of the surface k_l in the direction (l) along the boundary does not change at each point of the boundary, then the bending of the surface reduces to a motion.

The special case $\varphi(s) \equiv 0$ for surfaces of class C^3_α, $\alpha > 0$, was considered in ([2]).

To prove the theorem, we write Euler' s formula in the following form:

$$k_l = \frac{1}{k_n}(k_n\cos\varphi + \tau_g\sin\varphi)^2 + \frac{K}{k_n}\sin^2\varphi. \tag{6}$$

Hence, taking into account that under bending the normal curvature k_l does not change, for the function $W(z)$ we obtain a boundary condition of the form (2):

$$[-k_l + 2(k_n\cos\varphi + \tau_g\sin\varphi)\cos\varphi]\Delta k_n + 2(k_n\cos\varphi + \tau_g\sin\varphi)\sin\varphi\,\Delta\tau_g +$$
$$+ (\Delta k_n\cos\varphi + \Delta\tau_g\sin\varphi)^2 = 0,$$

where Δk_n and $\Delta\tau_g$ are determined by formulas (4).

According to condition (5), the index of the boundary condition is equal to -2. Consequently, the boundary-value problem (1), (6) has no solutions in a neighborhood of zero except the trivial one, which means that the surface is unbendable.

It also follows from (6) that if, under bending of the surface, the increment of the product $k_n \cdot k_l$ is equal to zero and

$$\frac{1}{2\pi}\Delta_L \varphi(s) > 2(m-1),$$

then the bending of the surface reduces to a motion.

4. In ([4]) Grotemeyer showed that a simply connected piece of a surface of positive curvature of class C^3 is unbendable with preservation of the "shadow" property of the boundary. The proof was carried out by the method of integral formulas. We shall show that this assertion remains valid under the weaker smoothness conditions indicated in § 1. Indeed, according to ([4]), under the assumptions made, along the boundary the condition

$$k_n \Delta\tau_g - \tau_g \Delta k_n = 0 \tag{7}$$

must be satisfied. Substituting (4) into (7), we obtain a boundary condition of the form (2) with index −2. Hence, by virtue of Theorem 1, the unbendability of the surface follows.

In conclusion, we note that with the aid of Theorem 1 one can prove a number of other sufficient criteria for unbendability of surfaces with boundary, most of which are transfers of the corresponding criteria of rigidity. For example, for the case $m = 0$, the invariance under bending of the spherical curvature of the boundary, of the angle θ between the principal normal to the boundary curve and the normal to the surface along it, of the curvature K' in the direction of the boundary curve, etc.

I express my deep gratitude to G. S. Barkhin for supervising this work.

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Received
2 VIII 1961

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Note: Figure translations are in progress. See original paper for figures.