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Abstract

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MATHEMATICS

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ON SUBDIRECTLY INDECOMPOSABLE SEMIGROUPS

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The theory of subdirect products of abstract algebras, whose foundations were laid by G. Birkhoff, has proved to be an important instrument in ring theory and has made it possible to obtain many new results in this area. It seems natural to study subdirect products of semigroups; however, as far as the author knows, there are no works devoted to this question.

In this note subdirectly indecomposable semigroups are considered. The importance of this class of semigroups follows from Theorem 1, going back to G. Birkhoff. In the commutative case it is possible to give a description of the structure of such semigroups. We note that a description of the structure of subdirectly indecomposable rings was also given only in the commutative case⁽¹⁾. Throughout what follows we consider semigroups containing more than one element.

Let $\{G_i\}_{i \in I}$ be a family of semigroups. On the Cartesian product of the family of sets $\{G_i\}_{i \in I}$ one can define a binary associative operation

$$\{G_i\}_{i \in I} \cdot (h_i)_{i \in I} = (g_i h_i)_{i \in I}$$

($g_i h_i$ denotes the product of the elements g_i and h_i of the semigroup G_i), with respect to which the Cartesian product

$$\prod_{i \in I} G_i$$

will be a semigroup. This semigroup is called the (complete) direct product of the family of semigroups $\{G_i\}_{i \in I}$.

By pr_k we shall denote the projection of the semigroup

$$\prod_{i \in I} G_i$$

onto the semigroup G_k (i.e. the mapping assigning to the element $(g_i)_{i \in I}$ the element g_k).

A subsemigroup G of the semigroup

$$\prod_{i \in I} G_i$$

is called a subdirect product of the family of semigroups $\{G_i\}_{i \in I}$ if, for every i , $pr_i(G) = G_i$. One says that a semigroup S decomposes into a subdirect product of the family of semigroups $\{G_i\}_{i \in I}$ if there exists an isomorphism ρ of the semigroup S onto a subsemigroup of

$$\prod_{i \in I} G_i$$

such that the semigroup $\rho(S)$ is a subdirect product of the family of semigroups $\{G_i\}_{i \in I}$. In this case ρ is called a subdirect decomposition of the semigroup S . In the case when S is isomorphic to some semigroup from the family $\{G_i\}_{i \in I}$, the decomposition ρ is called trivial. A semigroup is called subdirectly indecomposable if every subdirect decomposition of it is trivial.

From the results of G. Birkhoff ⁽²⁾ two theorems follow:

Theorem 1. *Every semigroup can be decomposed into a subdirect product of a family of subdirectly indecomposable semigroups.*

Theorem 2. *A semigroup is subdirectly indecomposable if and only if in the set of all stable equivalence relations existing between the elements of this semigroup, distinct from the identity relation, there is a least element.*

An element g of a semigroup G will be called separating if the unique stable equivalence relation between elements

of the semigroup G for which $\{g\}$ will be an equivalence class, is Δ_G^* .

Theorem 3. *Every subdirectly irreducible semigroup contains at least two distinct separating elements.*

Proof. Let g be some fixed element of the semigroup G . Introduce between the elements of the semigroup G a binary relation $\varepsilon_{\{g\}}$, defining $(g_1, g_2) \in \varepsilon_{\{g\}}$ if and only if, for any x, y ,

$$xg_1y = g \leftrightarrow xg_2y = g. \quad (*)$$

Here x, y may take values in the semigroup G , and may also be empty symbols (⁽³⁾, p. 7). It is not hard to show that $\varepsilon_{\{g\}}$ will be the largest of the stable equivalence relations between the elements of the semigroup G for which $\{g\}$ will be an equivalence class. Therefore the element g will be separating if and only if $\varepsilon_{\{g\}} = \Delta_G$. Clearly,

$$\bigcap_{g \in G} \varepsilon_{\{g\}} = \Delta_G.$$

If the semigroup G is subdirectly irreducible, then by Theorem 2 it follows that, for some g_1 , $\varepsilon_{\{g_1\}} = \Delta_G$. But, as is easy to see,

$$\bigcap_{g \neq g_1} \varepsilon_{\{g\}} = \Delta_G.$$

Therefore there will be an element $g_2 \neq g_1$ such that $\varepsilon_{\{g_2\}} = \Delta_G$. Hence g_1 and g_2 are distinct separating elements of the semigroup G .

An ideal** of a semigroup containing exactly one element is called a *zero* ideal. It is clear that all ideals of a semigroup that does not contain a zero are nonzero. The least zero ideal of a semigroup is called the *kernel* of this semigroup. We shall denote the kernel of the semigroup G by K .

Theorem 4. *Every subdirectly irreducible semigroup has a kernel.*

Proof. From Theorem 3 it follows that in the subdirectly irreducible semigroup G there is a separating element g distinct from zero. Let W be an ideal of the semigroup G and let $g \notin W$. Then, as is not hard to verify, for any $w_1, w_2 \in W$, $(w_1, w_2) \in \varepsilon_{\{g\}} = \Delta_G$, whence it follows that W is a zero ideal. Therefore, if W is a nonzero ideal, then $g \in W$. Consequently, the intersection of all nonzero ideals of the semigroup G contains an element distinct from zero, i.e. is the kernel of the semigroup G .***

Let us note that *from the existence of a kernel there does not follow the subdirect irreducibility of a semigroup*, as is the case for rings.

Let e be a central idempotent of the semigroup G (i.e. an idempotent commuting with every element of the semigroup G). Construct a binary relation ε_e between the elements of the semigroup G , defining $(g_1, g_2) \in \varepsilon_e$ if and only if $g_1 e = g_2 e$. It is not hard to see that ε_e will be a stable equivalence relation.

Let E be the set of elements of the semigroup G for which e serves as an identity (i.e. $g \in E$ if and only if $ge = eg = g$). Clearly, E will be an ideal. Construct the equivalence relation

$$\varepsilon^E = (E \times E) \cup \Delta_G,$$

which, as is not hard to see, will be stable. Let $(g_1, g_2) \in \varepsilon_e \cap \varepsilon^E$. If $g_1 \in E$, then $g_2 \in E$, $(g_1, g_2) \in \varepsilon_e$, and $g_1 = g_1 e = g_2 e = g_2$. If $g_1 \notin E$, then $g_2 \notin E$ and $g_1 = g_2$, since $(g_1, g_2) \in \varepsilon^E$. Consequently,

$$\varepsilon_e \cap \varepsilon^E = \Delta_G.$$

Therefore, to each central idempotent e

* Δ_G denotes the identity binary relation between the elements of the set G .

** By an "ideal," here and everywhere below, is meant a two-sided ideal.

*** We have actually proved that if a semigroup contains a separating element distinct from zero, then it has a kernel, and that all separating elements of the semigroup are contained in its kernel.

there will correspond a decomposition of the semigroup G into the subdirect product of the semigroups G/ε_e and G/ε^E (2). This decomposition will be trivial if and only if $\varepsilon^E = \Delta_G$ or $\varepsilon_e = \Delta_G$, i.e., when e is the zero or the identity of the semigroup G . The decomposition constructed by us is, in a certain sense, a generalization of the Pierce decomposition of a ring.

We note that from the preceding arguments it follows:

Theorem 5. *A subdirectly indecomposable semigroup contains no central idempotents other than the zero and the identity.*

Remark. It is not hard to show that if a semigroup G contains no identity (zero) and G^* (G^0) is the semigroup obtained from G by adjoining to G an identity (zero) ((3), pp. 73-74), then the semigroups G and G^* (the semigroups G and G^0) simultaneously are, or are not, subdirectly indecomposable.

Theorem 6. *If the intersection of all right, left, and two-sided ideals of a subdirectly indecomposable semigroup not containing a zero is nonempty, then the given semigroup is a group.*

Proof. Let K be the nonempty intersection of all right, left, and two-sided ideals. K will be a group, and the identity e of this group will be a central idempotent of the semigroup G ((3), pp. 251-252). By Theorem 5, e is the identity of the semigroup G . Consequently, $G = K$.

Corollary 1. *If the kernel K of a subdirectly indecomposable semigroup G is a group, then $G = K$.*

Indeed, in this case K will be the intersection of the right, left, and two-sided ideals of the semigroup G ((3), p. 249); consequently, $G = K$, by the preceding theorem, since K is a nonzero ideal.

Corollary 2. *Every subdirectly indecomposable commutative semigroup containing no divisors of zero is a subdirectly indecomposable abelian group or a subdirectly indecomposable abelian group with an adjoined zero.*

Proof. In view of the remark to Theorem 5, it suffices to prove Corollary 2 only in the case when the subdirectly indecomposable commutative semigroup G contains a zero. Let G^+ be the set of all elements of the semigroup G distinct from zero. Since G contains no divisors of zero, G^+ will be a subsemigroup of the semigroup G . Let 0^+ be the zero of the semigroup G^+ . Consider the stable equivalence relations between elements of G : $\varepsilon_1 = (G^+ \times G^+) \cup \{(0, 0)\}$ and $\varepsilon_2 = \Delta_G \cup \{(0, 0^+) \times (0, 0^+)\}$. Their intersection will be the identity relation; hence, by Theorem 2, $\varepsilon_1 = \Delta_G$, i.e., G^+ is a one-element abelian group. In the case when G^+ contains more than one element, G^+ cannot contain zero. From the remark to Theorem 5 it follows that G^+ will be a subdirectly indecomposable semigroup. Since the semigroup G^+ is commutative, in G^+ the notions of right, left, and

two-sided ideals coincide. By Theorems 6 and 4, G^+ is a group. Consequently, G is a subdirectly indecomposable abelian group with an adjoined zero.

Subdirectly indecomposable abelian groups (with an adjoined zero or without it) will be called **semigroups of the first type**.

Commutative semigroups G with zero, containing a separating element $g \neq 0$ such that $G\{g\} = \{0\}$, will be called **semigroups of the second type**.

A **semigroup of the third type** is a commutative semigroup G with zero and identity, containing a separating element distinct from zero, and such an element g , distinct from zero, that $(G \setminus A)\{g\} = \{0\}$, where A is the set of all invertible elements of the semigroup G , which is a subdirectly indecomposable abelian group.

We note that *semigroups of the third type are a special case of Rees supergroups* (4). If an identity is adjoined to a semigroup of the second type, a semigroup of the third type is obtained.

From formula (*) it follows that the condition “there exists a separating element” can be written in the form of an elementary formula. Therefore all the conditions entering into the definition of a semigroup of the second type or of the third type, with the exception of the subdirect irreducibility of A , can be written in the form of elementary axioms.

Theorem 7. *A commutative semigroup is subdirectly irreducible if and only if it belongs to the first, second, or third type.*

We omit the proof of this theorem, which is simple in essence but rather long.

Let W be an ideal of the semigroup G . The factor semigroup of the semigroup G by the stable equivalence relation

$$\varepsilon^W = (W \times W) \cup \Delta_G$$

will be denoted by G_W . The cyclic group of order p^α (p prime, $\alpha = 0, 1, 2, \dots$) will be denoted by Z_{p^α} .

The commutative semigroup with elements $\{0, g, f, e\}$, where 0 is a zero, e is an identity, $g^2 = gf = 0$, $f^2 = g$, will be denoted by A_1 , and its subsemigroups $\{0, g, f\}$, $\{0, g, e\}$, $\{0, g\}$, $\{0, e\}$ will be denoted respectively by A_2, A_3, A_4, A_5 .

From Theorem 7 one can derive:

Corollary. *Every finite commutative subdirectly irreducible semigroup has the form $Z_{p^\alpha} \times A_i / Z_{p^\alpha} \times \{0\}$ ($i = 1, \dots, 5$) or Z_{p^α} , and is characterized up to isomorphism by the specification of two natural numbers p^α and i .*

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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