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Abstract

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MATHEMATICS

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On a Class of Singular Integral Equations with Shift

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Certain geometric problems on the gluing of surfaces of positive curvature lead to singular integral equations containing the conjugate values of the unknown function and of the singular integral of it, as well as the values of these objects at the point t_1, obtained from t by a homeomorphic mapping $t_1 = \alpha(t)$ of the contour of integration onto itself.

In the present note we announce results obtained in the study of the singular integral equation

$$A(t)\overline{\varphi(t)} + C(t)\frac{1}{\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{\tau - t} + B(t)\varphi[\alpha(t)] + \frac{D(t)}{\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{\tau - \alpha(t)} +$$

$$+ \int_L \overline{K_1(t,\tau)}\,\varphi(\tau)\,d\tau + \int_L K_2[\alpha(t),\tau]\varphi(\tau)\,d\tau = H(t). \qquad (1)$$

Here the transformation $\alpha(t)$ satisfies Carleman' s condition ([1]), $\alpha[\alpha(t)] \equiv t$, on the simple closed Lyapunov contour L; the functions $A(t)$, $B(t)$, $C(t)$, $D(t)$, and $\alpha'(t)$ satisfy a Hölder condition on L, with $\alpha'(t) \neq 0$; $K_1(t,\tau)$ and $K_2(t,\tau)$ are Fredholm kernels.

If $K_1(t,\tau) \equiv K_2(t,\tau) \equiv 0$, then equation (1) will be called characteristic.

Consider the following system of singular integral equations:

$$A(t)\rho_1(t) + B(t)\rho_2(t) - \frac{C(t)}{\pi i}\int_L \frac{\overline{\tau}'^2\rho_1(\tau)}{\overline{\tau} - \overline{t}}\,d\tau + \frac{\lambda D(t)}{\pi i}\int_L \frac{\alpha'(\tau)\rho_2(\tau)}{\alpha(\tau) - \alpha(t)}\,d\tau +$$

$$+ \int_L \overline{K_1(t,\tau)}\,\overline{\tau}'^2\rho_1(\tau)\,d\tau + \lambda\int_L K_2[\alpha(t),\alpha(\tau)]\alpha'(\tau)\rho_2(\tau)\,d\tau = H(t), \qquad (2)$$

$$\overline{B[\alpha(t)]}\rho_1(t) + \overline{A[\alpha(t)]}\rho_2(t) - \frac{\overline{D[\alpha(t)]}}{\pi i}\int_L \frac{\overline{\tau}'^2\rho_1(\tau)}{\overline{\tau}-\overline{t}}\,d\tau +$$

$$+\frac{\lambda\overline{C[\alpha(t)]}}{\pi i}\int_L \frac{\alpha'(\tau)\rho_2(\tau)}{\alpha(\tau)-\alpha(t)}\,d\tau + \int_L \overline{K_2(t,\tau)}\,\overline{\tau}'^2\rho_1(\tau)\,d\tau +$$

$$+\lambda\int_L \overline{K_1[\alpha(t),\alpha(\tau)]}\alpha'(\tau)\rho_2(\tau)\,d\tau = \overline{H[\alpha(t)]},$$

where $\lambda = 1$ if the homeomorphism $\alpha(t)$ preserves the orientation on L, and $\lambda = -1$ if $\alpha(t)$ reverses the orientation. It is not difficult to see that if $\varphi(t)$ is a solution of the integral equation (1), then the functions $\rho_1(t) = \overline{\varphi(t)}$ and $\rho_2(t) = \varphi[\alpha(t)]$ are a solution of system (2). Conversely, if $\rho_1(t), \rho_2(t)$ is any solution of system (2), then the functions

$$\rho_1^*(t) = \frac{1}{2}\{\rho_1(t) + \overline{\rho_2[\alpha(t)]}\}, \qquad \rho_2^*(t) = \frac{1}{2}\{\rho_2(t) + \overline{\rho_1[\alpha(t)]}\}$$

also constitute a solution of system (2), with $\rho_1^*(t) = \overline{\rho_2^*[\alpha(t)]}$. Then the function $\varphi(t) = \rho_1^*(t)$ gives a solution of equation (1). Consequently, the nonhomogeneous equation (1) and the nonhomogeneous system (2) are simultaneously solvable or unsolvable. In considering homo-

of equation (1) and of the homogeneous system (2), it turns out that they have the same number of linearly independent solutions, if linear independence of solutions for equation (1) is understood in the sense of combinations with real coefficients, and for system (2) with complex coefficients. The number of linearly independent solutions of the homogeneous system of singular equations adjoint to system (2) also turns out to be equal to the number of linearly independent solutions of the integral equation

$$\overline{A(t)}\,t'^2\omega(t) + \lambda B[\alpha(t)]\alpha'(t)\omega[\alpha(t)] - \frac{1}{\pi i}\int_L \frac{\overline{C(\tau)}\,\tau u'^2\omega(\tau)}{\tau - t}\,d\tau - \frac{\lambda}{\pi i}\int_L \frac{D[\alpha(\tau)]\alpha'(\tau)\omega[\alpha(\tau)]}{\tau - t}\,d\tau + \int_L \overline{K_1(\tau,t)}\,\omega(\tau)\,d\tau \tag{3}$$

adjoint to equation (1).

In the case $\lambda = 1$, system (2) belongs to the normal type ([2]), if

$$\Delta(t) = [A(t) + C(t)]\,[\overline{A[\alpha(t)]} - \overline{C[\alpha(t)]}] - [B(t) - D(t)]\,[\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}] \neq 0,$$

and in the case $\lambda = -1$ the conditions for system (2) to belong to the normal type take the form

$$\Delta_1(t) = [A(t) + C(t)]\,[\overline{A[\alpha(t)]} + \overline{C[\alpha(t)]}] - [B(t) + D(t)]\,[\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}] \neq 0,$$

$$\Delta_2(t) = [A(t) - C(t)]\,[\overline{A[\alpha(t)]} - \overline{C[\alpha(t)]}] - [B(t) - D(t)]\,[\overline{B[\alpha(t)]} - \overline{D[\alpha(t)]}] \neq 0.$$

Let k and k' denote, respectively, the numbers of linearly independent solutions of the homogeneous equations (1) and (3).

Theorem 1. *The difference between the numbers of linearly independent solutions of the homogeneous equations (1) and (3) is given by the formulas*

$$\begin{aligned} k - k' &= 2\ \mathrm{Ind}\,\Delta(t), && \text{if } \lambda = 1; \\ k - k' &= \mathrm{Ind}\,\Delta_1(t) - \mathrm{Ind}\,\Delta_2(t), && \text{if } \lambda = -1. \end{aligned} \tag{4}$$

Theorem 2. *For solvability of the nonhomogeneous equation (1), it is necessary and sufficient that the condition*

$$\mathrm{Re}\left\{\int_L H(t)\omega(t)\,dt\right\} = 0, \tag{5}$$

be satisfied, where $\omega(t)$ is any solution of the adjoint homogeneous equation (3).

We note that, for the case $\alpha(t) \equiv t$, Theorems 1 and 2 were obtained by L. G. Mikhailov ([3]).

Under certain additional conditions imposed on the coefficients of the characteristic integral equation (1), the latter admits a complete qualitative investigation.

The characteristic integral equation (1) is equivalent to the boundary-value problem on L

$$[A(t)+C(t)]\,\overline{\Phi^+(t)}+[C(t)-A(t)]\,\overline{\Phi^-(t)}+[B(t)+D(t)]\,\Phi^+[\alpha(t)]+[D(t)-B(t)]\,\Phi^-[\alpha(t)] = H(t), \tag{6}$$

where $\Phi^+(t)$ and $\Phi^-(t)$ are the limiting values of an integral of Cauchy type with density $\varphi(t)$, with $\Phi^-(\infty) = 0$.

Suppose that $a(t)$ preserves the orientation on L ($\lambda = 1$), and that the following conditions are satisfied on L:

$$[A(t) + C(t)]\,[\overline{A[\alpha(t)]} + \overline{C[\alpha(t)]}] = [B(t) + D(t)]\,[\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}],$$

$$[A(t) - C(t)]\,[\overline{A[\alpha(t)]} - \overline{C[\alpha(t)]}] = [B(t) - D(t)]\,[\overline{B[\alpha(t)]} - \overline{D[\alpha(t)]}], \tag{7}$$

$$\overline{A[\alpha(t)]}D(t) - \overline{C[\alpha(t)]}B(t) \neq 0. \tag{8}$$

Condition (8), provided at least one of the equalities (7) holds, is equivalent to the condition $\Delta(t) \neq 0$ on L. Using (7) and (8), we reduce the boundary-value problem (6) to an equivalent system of boundary-value problems of Carleman type

$$\Phi^+[\alpha(t)] = -\frac{A(t)+C(t)}{B(t)+D(t)}\,\overline{\Phi^+(t)} + \frac{[\overline{C[\alpha(t)]} - \overline{A[\alpha(t)]}]H(t) - [D(t)-B(t)]\overline{H[\alpha(t)]}}{2[\overline{A[\alpha(t)]}D(t) - \overline{C[\alpha(t)]}B(t)]}, \tag{9}$$

$$\Phi^-[\alpha(t)] = -\frac{A(t)-C(t)}{B(t)-D(t)}\,\overline{\Phi^-(t)} + \frac{[\overline{C[\alpha(t)]} + \overline{A[\alpha(t)]}]H(t) - [D(t)+B(t)]\overline{H[\alpha(t)]}}{2[\overline{A[\alpha(t)]}D(t) - \overline{C[\alpha(t)]}B(t)]}. \tag{10}$$

The solution of problems (9) and (10) is given in our paper ([5]). It is easy to show, using (7), that formula (4) for the index of equation (1) takes the form

$$I = \varkappa^+ - \varkappa^-, \tag{11}$$

where

$$\varkappa^+ = \operatorname{Ind}\frac{A(t)+C(t)}{B(t)+D(t)}, \qquad \varkappa^- = \operatorname{Ind}\frac{A(t)-C(t)}{B(t)-D(t)}.$$

From the analysis of problems (9) and (10) one obtains:

Theorem 3. *The number of linearly independent solutions of the characteristic homogeneous equation (1) is expressed by the formula*

$$N = \frac{1}{2}\big[\,|\varkappa^+ + 1| + |\varkappa^- + 1| + \varkappa^+ - \varkappa^-\,\big].$$

The corresponding nonhomogeneous equation is solvable when the

$$R = \frac{1}{2}\big[\,|\varkappa^+ + 1| + |\varkappa^- + 1| - \varkappa^+ + \varkappa^-\,\big]$$

solvability conditions (5) are satisfied.

Let us note some consequences following from formula (11) and Theorem 3.

1. If $\varkappa^+ \geqslant 0,\ \varkappa^- < 0$, then $N = \varkappa^+ - \varkappa^-,\ R = 0$. If, however, $\varkappa^+ < 0,\ \varkappa^- \geqslant 0$, then $N = 0,\ R = \varkappa^- - \varkappa^+$.

Thus, in the indicated cases the characteristic equation (1) behaves analogously to the well-known characteristic singular equation with Cauchy kernel (see, for example, ([4])).

2. If $\varkappa^+ \geqslant 0,\ \varkappa^- \geqslant 0$ ($\varkappa^+ < 0,\ \varkappa^- < 0$), then we have $N = \varkappa^+ + 1$ ($N = -\varkappa^- - 1$), $R = \varkappa^- + 1$ ($R = -\varkappa^+ - 1$), i.e., if the homogeneous equation (1) is nontrivially solvable, then the corresponding nonhomogeneous equation, generally speaking, is unsolvable.
3. In the case $\varkappa^+ = \varkappa^-$, for the integral equation (1) the Fredholm alternative is valid, with the first part of the alternative ($N = R = 0$) occurring in the unique case $\varkappa^+ = \varkappa^- = -1$.

Let us indicate the most interesting special types of the characteristic integral equation (1). The case in which $B(t) \equiv C(t) \equiv 0$ was considered by the authors in the paper ([5]). In this case conditions (7) and (8) take the form

$$A(t)\overline{A[\alpha(t)]} = D(t)\overline{D[\alpha(t)]}, \qquad A(t) \neq 0, \quad D(t) \neq 0.$$

Here

$$\varkappa^+ = \varkappa^- = \operatorname{Ind}\frac{A(t)}{D(t)},$$

and for the equation under study the Fredholm alternative ($I = 0$) is valid.

From Theorem 1 there follows a more general result. Indeed, in the present case $\Delta(t) = \overline{\Delta[\alpha(t)]}$, and therefore $I = 2\operatorname{Ind}\Delta(t) = 0$ regardless of whether the condition $A(t)\overline{A[\alpha(t)]} = D(t)\overline{D[\alpha(t)]}$ is satisfied or not.

Let us consider also the integral equation

$$A(t)\overline{\varphi(t)} - \overline{B}(t)\varphi[\alpha(t)] + B(t)\frac{1}{\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{\tau - t} + \frac{A(t)}{\pi i}\int_L \frac{\varphi(\tau)\,d\tau}{\tau - \alpha(t)} = H(t). \quad (12)$$

Conditions (7) and (8) for equation (12) take the form

$$A(t)\overline{B[\alpha(t)]} + \overline{A[\alpha(t)]}B(t) = 0, \qquad \overline{A[\alpha(t)]}A(t) + \overline{B[\alpha(t)]}B(t) \neq 0.$$

We have $\chi^+ = -\chi^-$,

$$I = 2\chi^+ = 2\operatorname{Ind}\frac{A(t)+B(t)}{A(t)-B(t)}.$$

From Theorem 3 it follows that $N = 2\chi^+, R = 0$, if $\chi^+ > 0$; $N = 1, R = 1$, if $\chi^+ = 0$; $N = 0, R = -2\chi^+$, if $\chi^+ < 0$.

We note that by the indicated method one can solve the characteristic integral equation (1) when only one of the conditions (7), for example the first, and condition (8) are satisfied; but then, in the case $\chi^+ < 0, \chi^- < 0$, the numbers N and R will depend on the rank of a certain system of linear algebraic equations.

Let now $\alpha(t)$ change the orientation on L $(\lambda = -1)$. Condition (8) is still satisfied, as well as the condition

$$[A(t) + C(t)]\,[\overline{A[\alpha(t)]} - \overline{C[\alpha(t)]}] = [\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}]\,[B(t) - D(t)]. \tag{13}$$

Here condition (8) is equivalent to $\Delta_1(t) \neq 0,\ \Delta_2(t) \neq 0$. With the aid of (13), the boundary-value problem (6) is reduced to a boundary-value problem of the type of the Haseman problem [6]

$$\Phi^+[\alpha(t)] = \frac{A(t) - C(t)}{B(t) + D(t)}\,\overline{\Phi^-(t)} +$$

$$+\frac{[A(t) + C(t)]\overline{H[\alpha(t)]} - [\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}]H(t)}{[A(t) + C(t)][\overline{A[\alpha(t)]} + \overline{C[\alpha(t)]}] - [B(t) + D(t)][\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}]}.$$

Denote

$$\chi = \operatorname{Ind}\frac{A(t) - C(t)}{B(t) + D(t)}.$$

Theorem 4. *The characteristic equation (1) with conditions (8) and (13) is unconditionally solvable for $\chi \leq 0$, and its general solution depends linearly on χ arbitrary constants; while for $\chi > 0$, for solvability of this equation it is necessary and sufficient that χ conditions of the form (5) be satisfied.*

Assume now that the condition

$$\overline{A[\alpha(t)]}D(t) - \overline{C[\alpha(t)]}B(t) = 0 \tag{14}$$

is satisfied. Here it is not hard to show, using (14), that from the conditions $\Delta_1(t) \neq 0,\ \Delta_2(t) \neq 0$ there follows the condition $\Delta(t) \neq 0$, and conversely. Consequently, if (14) is satisfied, then system (2) belongs to the normal type both when $\lambda \simeq -1$ and when $\lambda \simeq 1$. The boundary-value problem (6) in the case under consideration is equivalently reduced to the Riemann boundary-value problem

$$\Phi^+(t) = \frac{\overline{B[\alpha(t)]} - \overline{D[\alpha(t)]}}{\overline{B[\alpha(t)]} + \overline{D[\alpha(t)]}}\,\Phi^-(t) +$$

$$+\frac{[\overline{A[\alpha(t)]}+\overline{C[\alpha(t)]}]H(t)-[B(t)+D(t)]\overline{H[\alpha(t)]}}{[\overline{A[\alpha(t)]}+\overline{C[\alpha(t)]}][\overline{A(t)}+\overline{C(t)}]-[\overline{B[\alpha(t)]}+\overline{D[\alpha(t)]}][\overline{B(t)}+\overline{D(t)}]}.$$

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Note: Figure translations are in progress. See original paper for figures.