V. G. KADYSHEVSKY

1962

SovietRxiv

Abstract

Full Text

V. G. KADYSHEVSKY

ON DIFFERENT PARAMETRIZATIONS IN THE THEORY OF QUANTIZED SPACE-TIME

(Presented by Academician N. N. Bogolyubov, 23 VI 1962)

§ 1. In the present paper, which is a continuation of work ([1])*, questions are discussed that are connected with the ambiguity in the definition of the 4-momentum vector in a theory of the Snyder type ([2–4]). This ambiguity is due to the fact that on the five-dimensional hypersphere

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \varepsilon\eta_4^2 = -\varepsilon \tag{1}$$

(cf. formula (2.5) of ([1])), which models p-space in the theory, one can introduce an infinite number of relativistically covariant coordinate systems (p_0, p_1, p_2, p_3), each of which, in the transition to ordinary pseudo-Euclidean** p-space, becomes its Cartesian system (see also ([3])). In papers ([1–4]) the role of the 4-momentum P_m $(m = 0, 1, 2, 3)$ is played by the coordinates of the projections of the points of the hypersphere (1) onto the tangent plane $\eta_4 = 1$ (the center of projection coincides with the center of the hypersphere)***. With the aid of (1) it is easy to establish that

$$\eta_m = \frac{p_m}{\sqrt{1-\varepsilon p^2}}, \qquad \eta_4 = \frac{1}{\sqrt{1-\varepsilon p^2}} \quad (p^2 = p_0^2 - \mathbf{p}^2), \tag{2}$$

whence (see ([1]), 2.4)

$$p_m = \eta_m/\eta_4. \tag{3}$$

It also follows from (2) that the surface (1) is mapped not onto the whole plane $\eta_4 = 1$, but only onto the region $p^2 < 1$ for $\varepsilon = 1$ and $p^2 > -1$ for $\varepsilon = -1$.

By moving the projection plane parallel to itself and displacing the center of projection along the η_4 axis, we can find all relativistically covariant parametrizations of the hypersphere (1). In particular, this gives the parametrization corresponding to the well-known **stereographic** projection (projection from the pole $(0, 0, 0, 0, 1)$ onto the tangent plane $\eta_4 = -1$), and also the **orthogonal** parametrization (projection from the point $(0, 0, 0, 0, \infty)$ onto the equatorial plane $\eta_4 = 0$). In the first case we have

$$\eta_m = \frac{p_m}{\varepsilon p^2/4 - 1}, \qquad \eta_4 = \frac{\varepsilon p^2/4 + 1}{\varepsilon p^2/4 - 1}. \tag{4}$$

Correspondingly, in the second case

$$\eta_m = p_m, \qquad \eta_4 = \sqrt{1 + \varepsilon p^2}. \tag{5}$$

The geodesic, stereographic, and orthogonal projections are singled out in an evident way among all methods of mapping a sphere onto a plane,

---

* Everywhere in what follows such a system of units is used in which $\hbar = c = l = 1$. In this case, evidently, for the transition to the ordinary theory it is sufficient to regard all quantities of the dimension of energy-momentum as much smaller than unity.

** The pseudo-Euclidean character of p-space arising as $l \to 0$ is ensured by the identity of the signs before ε in the right- and left-hand sides of (1).

*** In geometry such a parametrization of the sphere is called **geodesic**.

and the other projections are, in a certain sense, intermediate between these three. Therefore we restrict ourselves to considering only the indicated projections, i.e., we assume that the 4-impulse p_m is determined either by relation (2), or by (4), or by (5).

It is easy to see that in the case (4) the surface (1) is mapped onto the whole plane $\eta_4 = -1$. If diametrically opposite points of the hypersphere are identified [1], then the points p_m and q_m of the projection plane, related by the inversion transformation $p_m = 4\varepsilon q_m/q^2$, must also be regarded as identical. As a result, the projection region takes the form $-4 \le p^2 \le 4$. In the case (5), the hypersphere (1) is projected into the region $p^2 \ge -1$ for $\varepsilon = 1$, or into the region $p^2 \le 1$ for $\varepsilon = -1$. When diametrically opposite points of (1) are identified, these regions become topologically equivalent four-dimensional Möbius sheets, since their boundaries belong to (1) (cf. § 2 of [1]).

Having relations (2), (4), (5), we can easily transfer all conclusions and formulas from [1,2,4] to the stereographic and orthogonal parametrizations. Thus, for example, the displacement transformation (2.9) from [1] in the coordinates (4) is written as follows:

$$p_n(+)k_n = \frac{(p_n + k_n)(1 - \varepsilon k^2/4) + k_n \varepsilon (p+k)^2/4}{(1 - \varepsilon p^2/4)(1 - \varepsilon k^2/4) + \varepsilon (p+k)^2/4} \equiv q_n, \tag{6}$$

if $q^2 \ge -4$ for $\varepsilon = 1$ or $q^2 \le 4$ for $\varepsilon = -1$, and

$$p_n(+)k_n = 4\varepsilon q_n/q^2 \tag{7}$$

for the remaining q^2 (it is assumed that $-4 \leq p^2 \leq 4$ and $-4 \leq k^2 \leq 4$). With the aid of (6) we find the relation

$$q^2 = \frac{(p+k)^2}{(1-\varepsilon p^2/4)(1-\varepsilon k^2/4)+\varepsilon(p+k)^2/4}. \tag{8}$$

It follows from (6), (7), and (8) that $-4 \leq (p(+)k)^2 \leq 4$. Correspondingly, in the coordinates (5) we shall have:

$$p_n(+)k_n = \sigma\left[p_n + k_n\left(\sqrt{1+\varepsilon p^2} + \frac{\varepsilon(pk)}{1+\sqrt{1+\varepsilon k^2}}\right)\right]$$

$$(pk = p_0k_0 - pk), \tag{9}$$

where

$$\sigma = \operatorname{sign}\left[\sqrt{1+\varepsilon p^2}\sqrt{1+\varepsilon k^2} + \varepsilon(pk)\right].$$

From (9) it follows that

$$1+\varepsilon(p(+)k)^2 = \left(\sqrt{1+\varepsilon p^2}\sqrt{1+\varepsilon k^2}+\varepsilon(pk)\right)^2 \geq 0. \tag{10}$$

Thus, the transformation (9) does not take the vector p_m outside the projection region.

We still define the coordinate operators x^n as infinitesimal displacement operators in the curved p-space, i.e., it is assumed that

$$\varphi(p(+)k) = (1 - ik^n_{nx})\varphi(p) \tag{11}$$

for small k_n. With the aid of (6) and (9), one can find the explicit form of x^n in the corresponding parametrizations (the functions $\varphi(p)$ are regarded as scalars)*:

$$x^n = i\left[\delta^n_m\left(1+\frac{\varepsilon p^2}{4}\right) - \frac{\varepsilon}{2}p_m p^n\right]\frac{\partial}{\partial p_m} \qquad \text{(coordinates (4))}; \tag{12}$$

$$x^n = i\sqrt{1+\varepsilon p^2}\,\frac{\partial}{\partial p_n} \qquad \text{(coordinates (5))}. \tag{13}$$

* The matrix term that arises in the operators x^n in the case when the wave function is a spinor is the same for all parametrizations and coincides with (4.8) from [1].

Knowing the metric of the hypersphere (1) and using relations (4) and (5), it is easy to calculate the volume element of the curved p-space in the cases under consideration (cf. (6.2) from (1))

$$d\Omega_p = \sqrt{-g}\, d^4p = \frac{d^4p}{(1-\varepsilon p^2)^4} \quad \text{(coordinates (4));} \tag{14}$$

$$d\Omega_p = \sqrt{-g}\, d^4p = \frac{d^4p}{\sqrt{1+\varepsilon p^2}} \quad \text{(coordinates (5)),} \tag{15}$$

where g is the determinant of the metric tensor of p-space in the coordinates (4) and (5).

It is clear that in all parametrizations the equality holds:

$$d\Omega_{p(+)k} = d\Omega_p. \tag{16}$$

At the same time $d\Omega_{k(+)p} \neq d\Omega_p$, since $k(+)p$ is not a transformation of the vector p under motions of a space of constant curvature.

The property of "right invariance" of $d\Omega_p$ under shifts may be made the basis for the definition of the convolution of functions $f_1(p)$ and $f_2(p)$ in curved p-space:

$$f_1(p) * f_2(p) = \int_\Omega f_1(q)\, f_2(-(q(-)p))\, d\Omega_p, \tag{17}$$

where Ω is the region onto which the hypersphere (1) is projected under the chosen parametrization. Making in (17) the change of variables $q(-)p = -q'$, which is equivalent to $q = -(q'(-)p)$, it is easy to verify the commutativity of the new convolution operation:

$$f_1 * f_2 = f_2 * f_1. \tag{18}$$

However, (17) does not possess the usual property of associativity.

Putting in (17) $f_1(p) = \delta(p)$, $f_2(p) = f(p)$, we shall have, taking (18) into account:

$$\delta(p) * f(p) = \sqrt{-g(0)}\, f(-(0(-)p)) = f(p) = \int \delta(q(-)p)\, f(q)\, d\Omega_q. \tag{19}$$

On the other hand, $\delta(q(-)p) = \delta(p(-)q)$, since if $q(-)p = 0$, then $p(-)q = 0$, while for all other p and q one always has $\delta(q(-)p) = \delta(p(-)q) = 0$. Therefore the function $\delta(p(-)q)$ is the analogue of the ordinary four-dimensional function $\delta(p - q)$. From (19) it follows that

$$\delta(p(-)q) = \frac{1}{\sqrt{-g(p)}}\, \delta(p - q). \tag{20}$$

The presence of the new volume element requires a new definition of the operation of differentiating functionals of functions defined in curved p-space. Namely, if $F[f(p)]$ is such a functional, then we shall always take

$$\delta F[f(p)] = \int_\Omega \frac{\delta F[f(p)]}{\delta f(q)}\, \delta f(q)\, d\Omega_q. \tag{21}$$

Putting in (21) $F[f] = f$, we find the equality

$$\delta f(p) = \int_\Omega \frac{\delta f(p)}{\delta f(q)}\, \delta f(q)\, d\Omega_q,$$

whence, on the basis of (19),

$$\frac{\delta f(p)}{\delta f(q)} = \delta(p(-)q). \tag{22}$$

It is not difficult to verify that the operation of differentiation defined by relations (21) and (22) possesses all the properties of ordinary functional differentiation.

§ 2. From what has been set forth it follows that to each parametrization of the curved p-space there corresponds its own form of the operation $(+)$, of the coordinates x^n, and of the volume element $d\Omega_p$. Consequently, the problem of correctly defining the 4-momentum p_m in this

space can be reduced to finding the "proper"form of the shift $p(+)k$, the "proper" x^n, etc. We encounter an analogous situation in the relativistic space of velocities, which carries Lobachevsky geometry. It is known that the group of motions of the space in this case is the Lorentz group. On the other hand, by virtue of the definition $v_\alpha = dx_\alpha/dt$ $(\alpha = 1, 2, 3)$, Lorentz transformations with respect to the components v_α must be fractional-linear. Therefore, in v-space of constant curvature a geodesic coordinate system is realized (see (2.9) of [1]).

Returning again to the curved p-space, we may note that, among all its relativistically covariant parametrizations, the parametrization (5) is closest to the coordinate system in ordinary momentum space. Namely, only in this case does the equality hold (see (13))

$$[t, \mathbf{p}] = 0, \tag{23}$$

which permits one to pass from the $(p_0, \mathbf{p})$-representation to the "mixed" $(t, \mathbf{p})$-representation*. This circumstance seems to us essential for the further development of the theory in the scheme under consideration (for example, for formulating the principle of causality) and may, in our opinion, serve as the basis for defining the 4-momentum vector of curved p-space by means of relation (5). In the next paper, devoted to the investigation of a model of field theory in this formalism, we shall define the 4-momentum precisely in this way. In doing so we shall put $\varepsilon = 1$, i.e. choose Ω in the form $p^2 \geqslant -1$, so that the theory will have no restriction on the masses of physical systems.

The author expresses his deep gratitude to Yu. A. Gol'fand for numerous fruitful discussions.

Joint Institute
for Nuclear Research

Received
19 VI 1962

REFERENCES

[1] V. G. Kadyshevsky, ZhETF, **41**, 1885 (1961).
[2] H. Snyder, Phys. Rev., **71**, 38 (1947).
[3] V. L. Averbackh, B. V. Medvedev, DAN, **54**, 41 (1949).
[4] Yu. A. Gol' fand, ZhETF, **37**, 504 (1959).

* Let us emphasize in particular that equality (23) is satisfied in the orthogonal parametrization because the vector $\mathbf{p}$, being, by virtue of (5), the 3-dimensional part of the 5-vector (η_m, η_4), is orthogonal to the plane (04), while t is the infinitesimal rotation operator in this plane. Obviously, no other relativistically covariant coordinate system possesses this property.

Note: Figure translations are in progress. See original paper for figures.