Automorphisms of Homogeneous Convex Cones

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Abstract

Full Text

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Automorphisms of Homogeneous Convex Cones

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A **convex cone** in the m-dimensional linear space R^m is any set V possessing the following properties:

$$\text{(V1) if } x \in V \text{ and } \lambda > 0, \text{ then } \lambda x \in V;$$

$$\text{(V2) if } x, y \in V, \text{ then } x + y \in V.$$

In order to avoid considering uninteresting cases, we shall also make the following additional assumptions:

$$\text{(V3) the set } V \text{ contains no line;}$$

$$\text{(V4) the set } V \text{ linearly generates } R^m.$$

Cones which are carried into one another by nonsingular linear transformations are called **isomorphic**.

Nonsingular linear transformations A of the space R^m for which $AV = V$ are called **automorphisms of the cone** V. They form a group, denoted by $\mathfrak{G}(V)$. The elements of the Lie algebra of the group $\mathfrak{G}(V)$ are called **differentiations of the cone** V. It may happen that the group $\mathfrak{G}(V)$ acts transitively on V; then the cone V is called **homogeneous**. A homogeneous cone necessarily turns out to be open.

In my note ([1]) a one-to-one correspondence was established between homogeneous convex cones, considered up to isomorphism, and nonassociative algebras of a special kind. The theory of these algebras, similar to the theory of the j-algebras of Pyatetskii-Shapiro ([2,3]), was developed by B. Yu. Weisfeiler and the author. Using these results, it proved possible to study deeply the structure of homogeneous convex cones. In the present note a universal construction of homogeneous convex cones will be given, and the algebra of differentiations of the cone will be described in terms of this construction.

The algebras considered below are assumed to be finite-dimensional, not necessarily associative algebras over the field of real numbers.

A **matrix algebra of rank** n is any algebra $\mathfrak{A}$, bigraded by subspaces $\mathfrak{A}_{ij}$ $(i, j = 1, 2, \dots, n)$ in such a way that $\mathfrak{A}_{ij}\mathfrak{A}_{jk} \subset \mathfrak{A}_{ik}$ and $\mathfrak{A}_{ij}\mathfrak{A}_{lk} = 0$ for $j \neq l$. An element A of such an algebra is conveniently represented by a matrix (a_{ij}), where a_{ij} is the projection of A onto $\mathfrak{A}_{ij}$. An **involution** of a matrix algebra $\mathfrak{A}$ is its involutory antiautomorphism $*$, for which $\mathfrak{A}_{ij}^* \subset \mathfrak{A}_{ji}$.

Let $\mathfrak{A}$ be a matrix algebra with involution. The subspaces $\mathfrak{A}_{ii}$ $(i = 1, 2, \dots, n)$ are subalgebras of the algebra $\mathfrak{A}$. Suppose that all these subalgebras are isomorphic to the algebra of real numbers and that multiplication by elements from $\mathfrak{A}_{ii}$, defined in the subspaces $\mathfrak{A}_{ij}$ and $\mathfrak{A}_{ji}$, coincides with multiplication by real numbers, defined in $\mathfrak{A}_{ij}$ and $\mathfrak{A}_{ji}$ as in linear spaces. Let $A = (a_{ij}) \in \mathfrak{A}$; identifying the diagonal elements of the matrix A with the corresponding real numbers, put

$$\operatorname{Sp} A = \sum_i a_{ii}.$$

We shall also introduce the following notation:

$$[AB] = AB - BA, \qquad [ABC] = A(BC) - (AB)C.$$

An algebra $\mathfrak{A}$ is called a **compact matrix algebra** if:

$$\left.\begin{aligned} &(\text{K1})\ \operatorname{Sp}[AB] = 0, \\ &(\text{K2})\ \operatorname{Sp}[ABC] = 0 \end{aligned}\right\} \quad \text{for all } A,\ B,\ C \in \mathfrak{A};$$

$$(\text{K3})\ \operatorname{Sp} AA^* > 0 \quad \text{for } A \neq 0.$$

Put $\mathfrak{T} = \sum_{i \le j} \mathfrak{A}_{ij}$; the elements of $\mathfrak{T}$ are written in the matrix representation as upper triangular matrices. A compact matrix algebra $\mathfrak{A}$ is called a T**-algebra** if

$$\left.\begin{aligned} &(\text{T1})\ [TUW] = 0, \\ &(\text{T2})\ [TUU^*] = 0 \end{aligned}\right\} \quad \text{for all } T,\ U,\ W \in \mathfrak{T}.$$

Let $\mathfrak{A}$ be a T-algebra of rank n, and let $i \to i'$ be some permutation of the indices $1, 2, \dots, n$. We change the grading of the algebra $\mathfrak{A}$, renaming the subspace $\mathfrak{A}_{ij}$ as $\mathfrak{A}_{i'j'}$. If the permutation $i \to i'$ is such that from $i < j$, $i' > j'$ it follows that $\mathfrak{A}_{ij} = 0$, then the corresponding **change of grading** is called **inessential** and, as is easy to see, again leads us to a T-algebra. Two T-algebras are called **isomorphic** if they have the same rank and, after some inessential change of grading, become isomorphic as bigraded algebras with involution.

A **differentiation** of a T-algebra is any linear mapping of it into itself which is a derivation of the multiplicative structure, preserves the grading, and commutes with the involution.

To each T-algebra $\mathfrak{A}$ there corresponds a homogeneous convex cone $V(\mathfrak{A})$ in the space

$$\mathfrak{X} = \{X \in \mathfrak{A} : \ X^* = X\}$$

of "Hermitian" matrices of the algebra $\mathfrak{A}$. Namely, in the set $\mathfrak{T}$ of triangular matrices of the algebra $\mathfrak{A}$ consider the domain $\mathfrak{J}$ consisting of matrices with positive elements on the diagonal. It is easy to see that the set $\mathfrak{J}$ is closed with respect to multiplication and is a connected Lie group (associativity follows from axiom (T1)). The role of the identity in the group $\mathfrak{J}$ is played by the identity matrix $E = (\delta_{ij})$. Define further a mapping $F : \mathfrak{T} \to \mathfrak{X}$ by the formula

$$F(T) = TT^*$$

and put $V(\mathfrak{A}) = F(\mathfrak{J})$. It is not difficult to see that $V(\mathfrak{A})$ is the one-to-one image of the group $\mathfrak{J}$ and contains the matrix E, together with some neighborhood of it, in the space $\mathfrak{X}$. To left translations on the group $\mathfrak{J}$ there correspond transformations

$$\rho(W) : \ UU^* \to (WU)(U^*W^*) \qquad (U,\ W \in \mathfrak{J}) \tag{1}$$

of the set $V(\mathfrak{A})$. Differentiating with respect to W and putting $W = E,\ dW = T \in \mathfrak{T}$, we find the corresponding infinitesimal transformations:

$$\mathrm{d}\rho(T) : \ UU^* \to (TU)U^* + U(U^*T^*).$$

Axiom (T2) shows that the transformation $\mathrm{d}\rho(T)$ is the restriction to $V(\mathfrak{A})$ of the infinitesimal linear transformation

$$D_T : \ X \to TX + XT^* \qquad (X \in \mathfrak{X}) \tag{2}$$

of the space $\mathfrak{X}$. Consequently, the transformations $\rho(W)$ are also restrictions to $V(\mathfrak{A})$ of certain linear transformations of the space $\mathfrak{X}$.

Theorem 1. *For every T-algebra $\mathfrak{A}$, the set $V(\mathfrak{A}) \subset \mathfrak{X}$ is a homogeneous convex cone on which the group $\mathfrak{J}$ acts simply transitively and linearly by formula (1). All homogeneous convex cones are obtained*

In this way, with isomorphic cones corresponding to equivalent T-algebras, and only to them.

In every convex cone $V \subset R^m$ there is a canonical Riemannian metric (4), invariant with respect to all its automorphisms. The torsion-free linear connection generated by this metric is also invariant with respect to all automorphisms of the cone V. We shall denote the object of this connection by Γ. On the other hand, by their very definition the automorphisms of the cone V preserve the (flat) linear connection of the ambient vector space R^m, whose object we shall denote by $\overset{0}{\Gamma}$. The difference $\Gamma - \overset{0}{\Gamma} = P$ is a tensor field invariant with respect to all automorphisms of the cone V. Let $x_0 \in V$. The tensor $P(x_0)$, twice covariant and once contravariant, defines in the tangent space to V at the point x_0 the structure of a commutative algebra. By means of the natural identification this structure is transferred to the ambient space R^m. The resulting algebra is called the **connection algebra of the cone V at the point x_0**. It is clear that the automorphisms of the cone V leaving the point x_0 fixed are also automorphisms of the connection algebra at the point x_0. The converse, generally speaking, is false.

Theorem 2. *For the homogeneous convex cone $V(\mathfrak{A})$ corresponding to the T-algebra $\mathfrak{A}$, the structure of the connection algebra at the point $E = (\delta_{ij})$ is given in the space $\mathfrak{X}$ by the formula*

$$X \circ Y = \frac{1}{2}(XY + YX).$$

Let us now consider the question of finding the group $\mathcal{G}(V(\mathfrak{A})) = \mathcal{G}$. We already know its subgroup $\mathcal{T}$, acting simply transitively in $V(\mathfrak{A})$. If $\mathcal{H}$ is the group of all automorphisms of the cone $V(\mathfrak{A})$ preserving the matrix E, then there is the obvious decomposition $\mathcal{G} = \mathcal{T}\mathcal{H}$. Thus the matter reduces to finding the compact group $\mathcal{H}$. Its Lie algebra H will be described below.

According to what has been said above, the elements of the group $\mathcal{H}$ must be automorphisms of the connection algebra described in Theorem 2, and the elements of the Lie algebra H, consequently, must be derivations of this algebra. This serves as the starting point for finding them.

There are two ways of obtaining elements of the algebra H. The first consists in taking the restriction to $\mathfrak{X}$ of a derivation of the T-algebra $\mathfrak{A}$ (see above). The transformations thus obtained, as is easy to see, are contained in H and form a Lie algebra. We shall denote this linear Lie algebra by H_0.

The second way is the following. Let $K \in \mathfrak{A}$ be a "skew-Hermitian" matrix, i.e. $K + K^* = 0$. Consider the transformation

$$D_K : X \to [KX] \qquad (X \in \mathfrak{X}) \tag{3}$$

of the space $\mathfrak{X}$. It may happen that $D_K \in H$. The skew-Hermitian matrices for which this is satisfied form a subspace $\mathfrak{K}$ in the space of all skew-Hermitian matrices. The transformations D_K corresponding to them form in the Lie algebra H a subspace, which we shall denote by H_K.

Theorem 3. *The Lie algebra H decomposes into the direct sum of subspaces:*

$$H = H_0 + H_K,$$

and

$$[H_0, H_0] \subset H_0, \quad [H_0, H_K] \subset H_K.$$

The description of H_0, or, what is the same, the finding of all derivations of the T-algebra $\mathfrak{A}$, is a most interesting problem, but as yet unsolved. By contrast, H_K has been found completely. It is clear that H_K may be considered known if the space $\mathfrak{K}$ is known (see above). To formulate the answer, we shall need some preliminary constructions.

Let $\mathfrak{A}$ be a T-algebra of rank n. Put $N = \{1, 2, \dots, n\}$, $m_{ij} = \dim \mathfrak{A}_{ij}$. From the properties of the involution it follows that $m_{ij} = m_{ji}$. We shall consider

consider various equivalence relations in the set $\mathbf{N}$. We order these relations partially by setting $R_1 \ll R_2$ if $i \equiv j \pmod{R_1}$ implies $i \equiv j \pmod{R_2}$. First consider the following equivalence relation R_0 in the set $\mathbf{N}$: $i \equiv j \pmod{R_0}$, if $m_{ik} = m_{jk}$ for all $k \neq i, j$. Further, let $\mathfrak{R}_0$ be the set of all equivalence relations R in $\mathbf{N}$ satisfying the following two conditions:

$$\begin{aligned} &\text{(E1)} \quad R \ll R_0; \\ &\text{(E2)} \quad \text{if } i < j < k \text{ and } i \equiv k \pmod{R}, \end{aligned}$$

then either $m_{ij} = m_{jk} = 0$, or $i \equiv j \equiv k \pmod{R}$.

It turns out that in the set $\mathfrak{R}_0$ there exists a greatest element $\overline{R}$. Denote by M the set of pairs $\{i, j\}$ for which $i \equiv j \pmod{\overline{R}}$.

Theorem 4. *The space $\mathfrak{K}$ is formed by all skew-Hermitian matrices of the algebra $\mathfrak{A}$ contained in the subspace*

$$\sum_{\{i,j\} \in M} \mathfrak{A}_{ij}.$$

Remark. By an inessential change of the grading of the algebra $\mathfrak{A}$, one can arrange that the relation $\overline{R}$ satisfies the following condition, stronger than (E2):

$$(\overline{\text{E}}3) \quad \text{if } i < j < k \text{ and } i \equiv k \pmod{\overline{R}}, \text{ then } i \equiv j \equiv k \pmod{\overline{R}}.$$

In this case the matrices from $\mathfrak{K}$ will be brought to block-diagonal form.

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named after M. V. Lomonosov

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CITED LITERATURE

[1] E. B. Vinberg, DAN, **141**, No. 3 (1961). [2] I. I. Pyatetskii-Shapiro, DAN, **141**, No. 2 (1961). [3] I. I. Pyatetskii-Shapiro, Izv. AN SSSR, ser. matem., **26**, No. 1 (1962). [4] E. B. Vinberg, DAN, **133**, No. 1 (1960).

Note: Figure translations are in progress. See original paper for figures.