Mathematical Physics

B. V. MEDVEDEV and M. K. POLIVANOV

1962

SovietRxiv

Abstract

Full Text

Mathematical Physics

B. V. MEDVEDEV and M. K. POLIVANOV

ON THE ROLE OF COUNTERTERMS IN THE DISPERSION APPROACH TO QUANTUM FIELD THEORY

(Presented by Academician N. N. Bogolyubov, 28 XI 1961)

1. As is well known, in constructing quantum field theory one usually encounters divergent expressions; when they are eliminated there arises an element of ambiguity connected with the possibility of adding arbitrary counterterms (see, for example, ([1])). This circumstance is often perceived as a shortcoming of the theory, without which the theory would benefit greatly.

Recently we proposed ([2]) a different point of view on the role of counterterms. Namely, we expressed the opinion that the very possibility of obtaining nontrivial solutions in quantum field theory is opened only at the cost of the necessity, connected with infinities, of adding counterterms. The purpose of the present note is to illustrate this point of view by the example of counterterms associated with mass renormalization.

2. We shall work within the framework of the dispersion approach ([2–4]), since in this way it is possible to avoid the questionable aspects connected with perturbation theory, and we shall proceed from the infinite system of coupled equations studied in ([2]):

$$I(p,(p)_l;(q)_s) = P\left(\frac{q_1}{(q)_{s-1}}\right)\sqrt{2p^0\,2q_1^0}\,\delta(\mathbf{p}-\mathbf{q}_1)\,I((p)_l;(q)_{s-1}) - (2\pi)^{3/2}\sum_{\nu}^{\infty}\frac{1}{\nu!}\int\frac{(dk)_\nu}{(2k^0)_\nu}\,I((p)_l;(k)_\nu)\,I((k)_\nu;(q)_s) \times\left\{\frac{\delta(\mathbf{p}+\sum\mathbf{p}_i-\sum\mathbf{k}_\nu)}{\sum k_\nu^0-\sum p_i^0-p^0-i\varepsilon}-\frac{\delta(\mathbf{p}-\sum\mathbf{q}_j+\sum\mathbf{k}_\nu)}{-\sum k_\nu^0+\sum q_j^0-p^0-i\varepsilon}\right\}; \quad (1^1)$$

$$I((p)_l;q,(q)_s) = P\left(\frac{p_1}{(p)_{l-1}}\right)\sqrt{2p_1^0\,2q^0}\,\delta(\mathbf{p}_1-\mathbf{q})\,I((p)_{l-1};(q)_s)$$
$$-(2\pi)^{3/2}\sum_{\nu}^{\infty}\frac{1}{\nu!}\int\frac{(dk)_\nu}{(2k^0)_\nu}\,I((p)_l;(k)_\nu)\,I((k)_\nu;(q)_s)$$
$$\times\left\{\frac{\delta(-\mathbf{q}+\sum\mathbf{p}_i-\sum\mathbf{k}_\nu)}{\sum k_\nu^0-\sum p_i^0+q^0-i\varepsilon}-\frac{\delta(-\mathbf{q}-\sum\mathbf{q}_j+\sum\mathbf{k}_\nu)}{-\sum k_\nu^0+\sum q_j^0+q^0-i\varepsilon}\right\}. \qquad (1^2)$$

We shall use without explanation the notation of paper ([2]), with the improvement that, for brevity, a collection of symbols differing only in indices, for example $p_1,\ldots,p_l$, will now be written, as in formulas (1), by a single letter enclosed in parentheses, the index on which indicates the number of symbols: $(p)_l$.

3. Let us consider (1^1) for the function I with one argument (the propagation function):

$$I(p;-) = -(2\pi)^{3/2}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int\frac{(dk)_\nu}{(2k^0)_\nu}\,I(-;(k)_\nu)\,I((k)_\nu;-)$$
$$\times\left\{\frac{\delta(\mathbf{p}-\sum\mathbf{k}_\nu)}{\sum k_\nu^0-p^0-i\varepsilon}-\frac{\delta(\mathbf{p}+\sum\mathbf{k}_\nu)}{-\sum k_\nu^0-p^0-i\varepsilon}\right\}, \qquad (2)$$

and assume that no counterterms are added to the right-hand side of this equation (such an assumption will be consistent if all integrals on the right-hand side converge). By virtue of invariance with respect to three-dimensional rotations, the functions $I(-;k_1,\ldots,k_\nu)$ can, obviously, depend only on the scalar products $k_i\cdot k_j$ and, consequently, do not change under the replacement $k_i\to-\mathbf{k}_i$. Therefore

$$I(p;-) = -(2\pi)^{3/2}\int\frac{dk}{2k^0}\,|I(k;-)|^2\delta(\mathbf{p}-\mathbf{k})\frac{2k^0}{(k^0)^2-(p^0+i\varepsilon)^2}$$
$$-(2\pi)^{3/2}\sum_{\nu=2}^{\infty}\frac{1}{\nu!}\int\frac{(dk)_\nu}{(2k^0)_\nu}\,|I(k_1,\ldots,k_\nu;-)|^2\frac{2\sum k_\nu^0}{(\sum k^0)^2-(p^0+i\varepsilon)^2}\delta\big(\mathbf{p}-\sum\mathbf{k}_\nu\big)\,. \qquad (3)$$

From the stability condition for one-particle states it follows, as is easy to verify (see ([3]), § 4.1), that

$$I(p;-)=0;\qquad I(-;q)=0. \qquad (4)$$

Therefore the first line in (3) vanishes, and the condition imposed by this equation reduces to the requirement that the sum in the second line be equal to zero. Introducing the masses of the intermediate states

$$M_\nu^2 = \left(\sum k_\nu^0\right)^2 - \left(\sum \mathbf{k}_\nu\right)^2, \tag{5}$$

we transform the denominators in this sum to invariant form, after which (3) reduces to the requirement

$$(2\pi)^{3/2} \sum_{\nu=2}^{\infty} \frac{1}{\nu!} \int \frac{(dk)_\nu}{(2k^0)_\nu}\, 2 \sum k_\nu^0\, \delta\left(\mathbf{p} - \sum \mathbf{k}\right) \frac{|I((k)_\nu; -)|^2}{p^2 - M_\nu^2 + 2i\varepsilon p^0} = 0. \tag{6}$$

But the vector p lies on the energy surface (cf. ([2])), $p^2 = m^2$; as for the masses of the intermediate states M_ν, each of them satisfies $M_\nu \geq \nu m$, and since the sum in (6) begins with $\nu = 2$, all $M_\nu > m$, and consequently

$$\frac{1}{p^2 - M_\nu^2 + 2i\varepsilon p^0} = \frac{1}{m^2 - M_\nu^2} < 0. \tag{7}$$

All the remaining factors on the left-hand side of (6) are manifestly positive. Thus the sum on the left-hand side of (6) is a sum of essentially negative quantities, and from its being equal to zero it follows that each term separately is equal to zero. Consequently, we have shown that

$$I(k_1, \dots, k_\nu; -) = 0; \qquad I(-; k_1, \dots, k_\nu) = 0, \qquad \nu = 2, 3, \dots, \tag{8}$$

i.e. that from the infinite system of coupled equations (1) and the stability condition for one-particle states, in the absence of counterterms of the mass-renormalization type, there follows the vanishing of all matrix elements of the structure $I((p)_l; -)$ or $I(-; (q)_l)$.

4. Arguments of the type given above are by no means new and have been used to justify far-reaching conclusions ([6]) about the structure of quantum field theory. We shall now show, however, that along this path one can advance still further and, still assuming that counterterms are absent, prove the vanishing of all matrix elements, i.e. prove the theorem on the absence of any interaction:

$$S = 1. \tag{9}$$

Indeed, considering in the system ([12]) the equation for the function with one right argument,

$$I((p)_l;q) = P\left(\frac{p_1}{(p)_{l-1}}\right)\sqrt{2p_1^0\,2q^0}\,\delta(\mathbf{p}_1-\mathbf{q})\,I((p)_{l-1};-)$$
$$-(2\pi)^{3/2}\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int\frac{(dk)_\nu}{(2k^0)_\nu}\left\{\frac{\delta(-\mathbf{q}+\sum\mathbf{p}_i-\sum\mathbf{k}_\nu)}{\sum k_\nu^0-\sum p_i^0+q^0-i\varepsilon}+\frac{\delta(-\mathbf{q}-\sum\mathbf{k}_\nu)}{-\sum k_\nu^0+q^0-i\varepsilon}\right\}$$
$$\times I((p)_l;(k)_\nu)\,I((k)_\nu;-), \tag{10}$$

we see that, by virtue of (8), each of the terms on the right-hand side tends to zero, and, consequently, in the absence of counterterms all matrix elements with one right-hand argument will also be equal to zero

$$I(p_1,\dots,p_l;\ q_1)=0. \tag{11}$$

Supposing now that the equality to zero of the matrix elements with a number of right-hand arguments not exceeding n has already been proved, and considering equation (1^2) for the function with $n+1$ right-hand arguments, we convince ourselves that such matrix elements too must tend to zero. Thus the assertion that all matrix elements are equal to zero,

$$I(p_1,\dots,p_l;\ q_1,\dots,q_s)=0 \tag{12}$$

has been proved by induction. It is easy to see that assertion (9) follows from it.

5. The seeming paradoxical character of the result obtained has a simple explanation. For the foregoing arguments the assumption of the absence of counterterms is quite essential. It would be justified if all the integrals entering into the system (1) were convergent. Therefore, from the above rigorous result it follows that a nontrivial local theory without divergences is impossible—a result that has already been obtained ([7]) by a closely related method—and for the description of nature within the framework of a local scheme we are left only with theories containing divergences. For such theories our chain of proofs is immediately destroyed. Indeed, if in one of the equations of the system (1) we encounter a divergent integral, then this, as always (cf. ([3])), means that additional powers of the momentum should have been introduced into the denominator and an arbitrary counterterm of the required type added. In this case the arguments restricting the sign of the integral, if it were convergent, in no way extend to the counterterm. Therefore from condition (6) one can now extract only an indication of the sign of the mass counterterm.

For a complete substantiation of the point of view on counterterms proposed by the authors in ([2]), it would still be necessary to show that even in the presence of a mass counterterm one cannot go beyond the trivial, if in the theory there is not at least one counterterm of the charge type at one of the higher vertices. We expect to return to this question.

6. Let us make one more remark. In the derivation given above one could reproach us for basing ourselves on the system (1), which itself, since it uses Fourier transforms of ϑ-functions without due justification, does not belong to the class of rigorous results of the dispersion method ([5]). However, we have resorted to this instrument only for the sake of making the derivation more transparent. In fact, for the proof of the central point —the proposition (6)—(8)—there is essentially no need for any system of coupled equations at all; it is sufficient to have the Jost-Lehmann spectral representation, which is established quite rigorously ([3]).

Indeed, by the definition of I:

$$I(p;\ -) = \sqrt{2p^0}\,\langle p|j(0)|0\rangle = \frac{1}{(2\pi)^{3/2}} \int dx\, e^{ipx} \left\langle 0 \left| \frac{\delta i(0)}{\delta\varphi(x)} \right| 0 \right\rangle , \tag{13}$$

or, if we use the functions $f^{\mathrm{ret}}(x)$ and $g^{\mathrm{ret}}(p)$ introduced in ([3]), § 4,

$$I(p;\ -) = -\frac{1}{(2\pi)^{3/2}} \int dx\, e^{ipx} f^{\mathrm{ret}}(x) = -\frac{1}{(2\pi)^{3/2}} g^{\mathrm{ret}}(p)\Big|_{p^0=+\sqrt{\mathbf{p}^2+m^2}} . \tag{14}$$

But for $g^{\mathrm{ret}}(p)$ the Jost-Lehmann representation holds (see ([3]), equation 4.38)

$$g^{\mathrm{ret}}(p) = (p^2-m^2)^{n+1} \int_{(2m)^2}^{\infty} \frac{\Upsilon(\zeta)\, d\zeta}{(\zeta-m^2)^{n+1}(\zeta-p^2-i\varepsilon p^0)} + \sum_{i=0}^{n} C_i (m^2-p^2)^i, \tag{15}$$

the spectral function $\Upsilon(\xi)$ of which is given by expression (4.7) from ([3]), being expressed in terms of our matrix elements I, has the form

$$\begin{aligned} \Upsilon(k^2) = (2\pi)^3 \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \int \frac{(dk)_\nu}{(2k^0)_\nu} \left|I(-;(k)_\nu)\right|^2 \delta\left(k - \sum_1^{\nu} k_\nu\right) \\ \times 2\sum_1^{\nu} \sqrt{k_i^2+m^2}\,\delta(k^2-M_\nu^2)\,. \end{aligned} \tag{16}$$

For the case of the absence of counterterms in (15), $n+1=0$, and, substituting (16) there, we arrive exactly at condition (6) and the consequences (8) following from it.

As for the further induction argument (equations (10)—(12)), it is not difficult to convince oneself that here, too, one can dispense with the Fourier transforms of

ϑ-functions, basing the reasoning not on system (1), but on the corresponding, more cumbersome system in the x-representation (see ([2]), equations (9) and (13)). Thus all the arguments can be given the required degree of rigor.

In conclusion, the authors express their gratitude to N. N. Bogolyubov for discussion of the results obtained.

Mathematical Institute named after V. A. Steklov
of the Academy of Sciences of the USSR

Joint Institute for Nuclear Research

Received
15 XI 1961

REFERENCES

[1] N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Moscow, 1957.
[2] B. V. Medvedev, M. K. Polivanov, ZhETF, **41**, 1130 (1961).
[3] N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov, *Problems in the Theory of Dispersion Relations*, Moscow, 1958.
[4] B. V. Medvedev, DAN, **135**, 1087 (1960); ZhETF, **40**, 826 (1960).
[5] N. N. Bogolyubov, V. S. Vladimirov, A. A. Logunov, B. V. Medvedev, M. K. Polivanov, B. M. Stepanov, Survey report at the IV All-Union Mathematical Congress, Leningrad, July 1961.
[6] R. Haag, Dan. Mat. Fys. Medd., **29**, No. 12 (1955).
[7] V. L. Bonch-Bruevich, B. V. Medvedev, ZhETF, **22**, 425 (1952).

Note: Figure translations are in progress. See original paper for figures.