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Abstract

Full Text

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DIFFERENTIAL PROPERTIES OF SOLUTIONS OF STABLE HYPOELLIPTIC EQUATIONS

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Let us consider the differential operator $\mathcal{P}\left(x, \dfrac{\partial}{\partial x}\right)$

$$\mathcal{P}\left(x, \frac{\partial}{\partial x}\right) = \sum_{\alpha_1 \dots \alpha_n} a_{\alpha_1 \dots \alpha_n}(x) \frac{\partial^{\sum_1^n \alpha_i}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

in a domain $D \subset R^{(n)}$. If the coefficients of the operator are sufficiently smooth functions, then the operator admits the representation

$$\begin{aligned}\mathcal{P}\left(x, \frac{\partial}{\partial x}\right) = {} & \sum_{i_1 \dots i_n} (-1)^{\sum_1^n i_k} \frac{\partial^{\sum_1^n i_k}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} a_{2i_1 \dots 2i_n}(x) \frac{\partial^{\sum_1^n i_k}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \\ & + \sum_{\alpha_{i_1} \dots \alpha_{i_n}, s} (-1)^{\beta_1} D^{\beta_1}_{1,\alpha_{i_1} \dots \alpha_{i_n}, s} c_{2\alpha_{i_1} \dots 2\alpha_{i_s}, 2\alpha_{i_{s+1}}+1, \dots, 2\alpha_{i_n}+1}(x) D^{\beta_2}_{2,\alpha_{i_1} \dots \alpha_{i_n}, s},\end{aligned} \tag{1}$$

where

$$D^{\beta_1}_{1,\alpha_{i_1} \dots \alpha_{i_n}, s} = \frac{\partial^{\beta_1}}{\partial x_{i_1}^{\alpha_{i_1}} \cdots \partial x_{i_s}^{\alpha_{i_s}} \cdots \partial x_{i_{s+k}}^{\alpha_{i_{s+k}}+1+k-2\left[\frac{k}{2}\right]} \cdots \partial x_{i_n}^{\alpha_{i_n}+1+n-s-2\left[\frac{n-s}{2}\right]}};$$

$$D^{\beta_2}_{2,\alpha_{i_1} \dots \alpha_{i_n}, s} = \frac{\partial^{\beta_2}}{\partial x_{i_1}^{\alpha_{i_1}} \cdots \partial x_{i_s}^{\alpha_{i_s}} \cdots \partial x_{i_{s+k}}^{\alpha_{i_{s+k}}+2\left[\frac{k}{2}\right]-k} \cdots \partial x_{i_n}^{\alpha_{i_n}+2\left[\frac{n-s}{2}\right]-(n-s)}};$$

$a_{2i_1 \dots 2i_n}(x) \geq k > 0$ for $x \in D$; $[k]$ is the least integer not less than k.

Let us denote

$$P_1\left(x, \frac{\partial}{\partial x}\right) = \mathcal{H}\left(x, \frac{\partial}{\partial x}\right) + Q\left(\frac{\partial}{\partial x}\right),$$

where

$$\mathcal{H}\left(x,\frac{\partial}{\partial x}\right)=\sum_{i_1\dots i_n}(-1)^{\sum_1^n i_k}\frac{\partial^{\sum_1^n i_k}}{\partial x_n^{i_1}\cdots\partial x_n^{i_n}}\,a_{2i_1\dots 2i_n}(x)\frac{\partial^{\sum_1^n i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}};$$

$$Q\left(\frac{\partial}{\partial x}\right)=\sum_{\alpha_{i_1}\dots\alpha_{i_n},\,s}\frac{\partial^{2\sum_1^n\alpha_{i_k}}}{\partial x_{i_1}^{2\alpha_{i_1}}\cdots\partial x_{i_n}^{2\alpha_{i_n}}}\left[\frac{\partial^{2\left(\beta_1-\sum_1^n\alpha_{i_k}\right)}}{\partial x_{i_{s+2}}^2\cdots\partial x_{i_{s+k}}^{2\left(1+k-2\left[\frac{k}{2}\right]\right)}\cdots\partial x_{i_n}^{2\left(1+n-s-2\left[\frac{n-s}{2}\right]\right)}}\right.$$
$$\left.+\frac{\partial^{2\left(\beta_2-\sum_1^n\alpha_{i_k}\right)}}{\partial x_{i_{s+1}}^2\cdots\partial x_{i_{s+k}}^{2\left(2\left[\frac{k}{2}\right]-k\right)}\cdots\partial x_{i_n}^{2\left(2\left[\frac{n-s}{2}\right]-n+s\right)}}\right].$$

Alongside the operators $P\left(x,\frac{\partial}{\partial x}\right)$, $P_1\left(x,\frac{\partial}{\partial x}\right)$, $\mathcal{H}\left(x,\frac{\partial}{\partial x}\right)$, we introduce into consideration the polynomials

$$\mathcal{H}(x,\sigma)=\sum_{i_1\cdots i_n}a_{2i_1\cdots 2i_n}(x)\sigma_1^{2i_1}\cdots\sigma_n^{2i_n};$$

$$P(x,\sigma)=\mathcal{H}(x,\sigma)+\sum_{\alpha_{i_1}\cdots\alpha_{i_n},\,s}c_{2\alpha_{i_1}\cdots 2\alpha_{i_s},\,2\alpha_{i_{s+1}}+1\cdots 2\alpha_{i_n}+1}(x)\times$$
$$\times\,\sigma_{i_1}^{2\alpha_{i_1}}\cdots\sigma_{i_s}^{2\alpha_{i_s}}\sigma_{i_{s+1}}^{2\alpha_{i_{s+1}}+1}\cdots\sigma_{i_n}^{2\alpha_{i_n}+1};$$

$$\mathcal{P}_1(x,\sigma)=\mathcal{H}(x,\sigma)+\sum_{\alpha_{i_1}\cdots\alpha_{i_n},\,s}\sigma_{i_1}^{2\alpha_{i_1}}\cdots\sigma_{i_n}^{2\alpha_{i_n}}\left[\sigma_{i_{s+2}}^2\cdots\sigma_{i_{s+k}}^{2\left(1+k-2\left[\frac{k}{2}\right]\right)}\cdots\right.$$
$$\left.\cdots\sigma_{i_n}^{2\left(1+n-s-2\left[\frac{n-s}{2}\right]\right)}+\sigma_{i_{s+1}}^2\cdots\sigma_{i_{s+k}}^{2\left(2\left[\frac{k}{2}\right]-k\right)}\cdots\sigma_{i_n}^{2\left(2\left[\frac{n-s}{2}\right]-n+s\right)}\right].$$

We shall call the operator $\mathcal{P}\left(x,\frac{\partial}{\partial x}\right)$ stable if the inequalities

$$\mathcal{H}(x,\sigma)+c\geq c_1\sigma_1^{2m_1}\cdots\sigma_n^{2m_n},\qquad c_1\geq 0,$$

$$c_1\int_D\sum_{i_1\cdots i_n}a_{2i_1\cdots 2i_n}(x)\left[\frac{\partial^{\sum_1^n i_k}\varphi}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\right]^2dx\geq\left|\int_D\left[\sum_{i_1\cdots i_n}a_{2i_1\cdots 2i_n}(x)\left[\frac{\partial^{\sum_1^n i_k}\varphi}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\right]^2\right.\right.$$
$$\left.\left.+\sum_{\alpha_{i_1}\cdots\alpha_{i_n},\,s}c_{2\alpha_{i_1}\cdots 2\alpha_{i_n}+1}(x)D^{\beta_1}_{1,\alpha_{i_1}\cdots\alpha_{i_n},s}\varphi\,D^{\beta_2}_{2,\alpha_{i_1}\cdots\alpha_{i_n},s}\varphi\right]dx\right|\geq\qquad(2)$$

$$\geq c_2 \int_D \sum_{i_1 \cdots i_n} a_{2i_1 \cdots 2i_n}(x) \left[\frac{\partial^{\sum_1^n i_k} \varphi}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \right]^2 dx, \qquad \varphi(x) \in C_0^\infty(D),$$

hold, where $\sigma_1^{2m_1} \cdots \sigma_n^{2m_n}$, up to a coefficient, is one of the terms of the polynomial $\mathcal{P}_1\left(x, \dfrac{\partial}{\partial x}\right)$.

We formulate the hypoellipticity condition $\left({}^{1,2}\right)$ for stable operators as follows. Let there exist numbers $\alpha_i^0 > 0$, $i = 1, \dots, n$, such that for all α_i, $0 \leq \alpha_i \leq \alpha_i^0$, the inequality

$$c_1 \mathcal{H}(x, \sigma) + c_1 \geq \left| \sigma_1^{2m_1 - 2k_1 + 2\alpha_1} \cdots \sigma_n^{2m_n - 2k_n + 2\alpha_n} \right|,$$

$$m_i \geq k_i, \qquad \sum_1^n k_i \geq 1, \quad i = 1, \dots, n, \tag{3}$$

is valid.

Let $u(x)$ be a generalized function of finite order of singularity p, $\left({}^3\right)$, and be a weak generalized solution of the equation $\mathcal{P}u = f$ in the domain D, i.e.

$$\left(u, \mathcal{P}^*\left(x, \frac{\partial}{\partial x}\right)\varphi\right) = (f, \varphi), \qquad f(x) \in L_2(D) \tag{4}$$

for all $\varphi(x)$ from $C_0^\infty(D)$. Consider $\psi(x)$, which is a solution of the equation $\Delta^q \psi = u$, where Δ is the Laplace operator. Then $\psi(x)$ has $2q - p$ continuous derivatives inside the domain D. Let the order of the operator $\mathcal{P}\left(x, \dfrac{\partial}{\partial x}\right)$ is equal to l. Choose q equal to $[l/2] + p$. Obviously, if the operator $\mathcal{P}\left(x, \dfrac{\partial}{\partial x}\right)$ satisfies conditions (1), (2), (3), then the operator $\Delta^q \mathcal{P}^*\left(x, \dfrac{\partial}{\partial x}\right)$ also satisfies the same conditions, provided only that the coefficients of the operator $\mathcal{P}\left(x, \dfrac{\partial}{\partial x}\right)$ are sufficiently smooth. Taking into account the smoothness of the function $\psi(x)$ and representation (1), we rewrite equality (4) in the form

$$\int_D \left[\sum_{i_1 \dots i_n} b_{i_1 \dots i_n}(x) \frac{\partial^{\sum i_k} \psi}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \frac{\partial^{\sum i_k} \varphi}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} + \sum_{\alpha_{i_1} \dots \alpha_{i_n}, s} d_{2\alpha_{i_1} \dots 2\alpha_{i_n} + 1}(x) D^{\beta_1}_{1, \alpha_{i_1} \dots \alpha_{i_n}, s} \psi D^{\beta_2}_{2, \alpha_{i_1} \dots \alpha_{i_n}, s} \varphi \right] dx = (f, \varphi \tag{5}$$

Consider the closure of the functions from $C^\infty(\overline{D})$ in the norm

$$\|u\|_{P_1} = \left\{ \int_D \sum_{i_1 \dots i_n} \left(\frac{\partial^{\sum_1{}^n i_k} u}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \right)^2 + \right.$$

$$\left. + \sum_{\alpha_{i_1} \dots \alpha_{i_n}, s} \left[\left(D^{\beta_1}_{1, \alpha_{i_1} \dots \alpha_{i_n}, s} u \right)^2 + \left(D^{\beta_2}_{2, \alpha_{i_1} \dots \alpha_{i_n}, s} u \right)^2 \right] dx \right\}^{1/2}. \qquad (6)$$

The closure of the functions thus obtained is a Hilbert space; denote it by $H_{P_1}(D)$. Under the closure of the functions from $C_0^\infty(\overline{D})$ in norm (6), one obtains a certain new Hilbert space $\overset{0}{H}_{P_1}(D)$, which obviously coincides with the space $\overset{0}{H}_H(D)$.

A function $u(x)$ from $H_{P_1}(D)$ will be called a strong generalized solution of the equation

$$\mathcal{P}u = f, \qquad f \in L_2(D), \qquad (7)$$

if $u(x)$ satisfies equality (5) for every $\varphi(x)$ from $\overset{0}{H}_H(D)$. It is easy to see that the function $\psi(x)$ considered above is a strong generalized solution of the equation

$$\Delta^q \mathcal{P}^* \left(x, \frac{\partial}{\partial x} \right) \psi = f.$$

Introduce the Hilbert space $H_{D^{2k_i}_{x_i}}(D)$, which is the closure of $C^\infty(\overline{D})$ in the norm

$$\left\{ \int_D \sum_{j=0}^{k_i} \left(\frac{\partial^j \varphi}{\partial x_i^j} \right)^2 dx \right\}^{1/2}. \qquad (8)$$

For strong generalized solutions of equations the following holds.

Theorem 1. *Let the coefficients of a stable hypoelliptic operator of order l, written in the form (1), belong to the space*

$$C^{[l/2]}(D) \bigcap_{i=1}^n C^{[l/2]+k_i}_{x_i}(D'),$$

$f(x)$ to the space

$$L_2(D) \bigcap_{i=1}^n H_{D^{2k_i}_{x_i}}(D').$$

Then $u(x)$ *belongs to the space*

$$H_{P_1}(D)\bigcap_{i=1}^{n} H_{P_1^1 D_{x_i}^{2k_i}}(D'), \qquad \overline{D'} \subset D.$$

Of interest is the study of the behavior of a strong generalized solution as x approaches the boundary of the domain. An answer to this question, although a very imprecise one, is given by the following theorem.

Theorem 2. If the coefficients of a stable hypoelliptic operator of order l, written in the form (1), belong to the space

$$C^{\left[\frac{l}{2}\right]+k}(\overline{D}),$$

and the right-hand side of the equation $f(x)$ is from the space

$$W_2^{(k)}(D) = \bigcap_{i=1}^{n} H_{Dx_i}^{2k}(D),$$

then

$$\rho(x,\dot{D})^{\frac{\left[\frac{l}{2}\right]+k}{\min_i a_i^0}\, l+\left[\frac{l}{2}\right]-1}\, u(x)$$

belongs to the space

$$\bigcap_{i=1}^{n} H_{\rho_1^2 Dx_i}^{2k}(D) \subset W_2^{2+k}(D),$$

where $\rho(x,\dot{D})$ is the distance from the point x to the boundary of the domain D.

Using the embedding theorems (4), namely

$$W_2^{2+k}(D) \subset C^{\,1-k-\left[\frac{n}{2}\right]}(\overline{D}),$$

we obtain

$$\rho(x,\dot{D})^{\frac{\left[\frac{l}{2}\right]+k}{\min_i a_i^0}\, l+\left[\frac{l}{2}\right]-1}\, u(x) \in C^{\,1-k-\left[\frac{n}{2}\right]}(\overline{D})$$

or

$$\rho(x,\dot{D})^{\frac{\left[\frac{l}{2}\right]+k}{\min_i a_i^0}\, l+\left[\frac{l}{2}\right]-1}\, D^{\,1+k-\left[\frac{n}{2}\right]}u(x) \in C(\overline{D}).$$

Thus, as the point x approaches the boundary of the domain D, a strong generalized solution and all its derivatives, if the coefficients and the right-hand side of the equation are infinitely differentiable, can grow no faster than some power function.

The method of proof of the theorems indicated above essentially coincides with the scheme developed in (5), but requires the use of a more delicate analytic apparatus; namely, in the present work the concept of a derivative of fractional order plays an essential role.

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Note: Figure translations are in progress. See original paper for figures.

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