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## Abstract

## Full Text

A. A. TEMLYAKOV

# INTEGRAL REPRESENTATIONS FOR MERO-MORPHIC FUNCTIONS

(Presented by Academician V. I. Smirnov on 23 XI 1961)

1. In papers <sup>(1-3)</sup> I established that if a function  $F(w, z)$  is regular in a bounded complete doubly circular domain  $D \ni (0, 0)$  with twice continuously differentiable and analytically convex exterior defining boundary hypersurface  $\Gamma$  (i.e., in the domain  $D$ , bounded by the hypersurfaces  $\Gamma_1\{0 \leq |w| \leq r_1(0), |z| = r_2(0)\}$ ,  $\Gamma\{|w| = r_1(\tau), |z| = r_2(\tau), 0 < \tau < 1\}$ ,  $\Gamma_2\{|w| = r_1(1), 0 \leq |z| \leq r_2(1)\}$ , where  $r_1(\tau)$  is a positive, continuously differentiable function on the interval  $0 < \tau < 1$ , satisfying the conditions  $r_1(0) = \lim_{\tau \rightarrow 0} r_1(\tau) \geq 0$ ,  $0 \leq r_1'(\tau) \leq \frac{r_1(\tau)}{\tau}$

and

$$r_2(\tau) = R_2 \exp \left[ - \int_0^\tau \frac{\tau}{1-\tau} d \ln r_1(\tau) \right],$$

and if  $L[F] = F + wF_w + zF_z$  is a continuous function in the closed domain  $\bar{D}$ , then for  $(w_0, z_0) \in D$

$$F(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{L^{(\Gamma)}[F]}{\zeta - u_0} d\zeta, \quad (1)$$

where

$$u_0 = \tau \frac{w_0}{r_1(\tau)} + (1 - \tau) \frac{z_0}{r_2(\tau)} e^{it},$$

and  $L^{(\Gamma)}[F]$  is the value of the operator  $L(F)$  on the hypersurface  $\Gamma\{w = r_1(\tau)\zeta, z = r_2(\tau)\zeta e^{-it}\}$ . Formula (1) expresses  $F(w, z)$  inside the domain  $D$  in terms of the values of the linear differential operator  $L[F]$  on the part  $\Gamma$  of the boundary of the domain  $\bar{D}$ .

In the present note formula (1) is generalized to meromorphic functions  $F(w, z) = \varphi(w, z)/f(w, z)$ , where  $\varphi(w, z)$ ,  $f(w, z)$  are regular functions in the domain  $D$  and  $f(w, z) \not\equiv 0$ , and also to functions of a more general nature.

2. First consider functions of the form  $F(w, z) = \varphi(w, z)/f(w/z)$ , where  $f(\zeta)$  is an entire function.

**Theorem 1.** If the product  $f(w/z)F(w, z)$  is a regular function in the domain  $D$  and the operator  $L[f(w/z)F(w, z)]$  is continuous in the closed domain  $\bar{D}$ , then for  $(w_0, z_0) \in D$ , but  $f(w_0/z_0) \neq 0$ ,

$$F(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} K(w, z; w_0, z_0) \frac{L^{(\Gamma)}[F]}{\zeta - u_0} d\zeta, \quad (2)$$

where

$$K(w, z; w_0, z_0) = f(w/z) : f(w_0/z_0), \quad (w, z) \in \Gamma.$$

**Proof.** Since the operator

$$\omega \left[ f \left( \frac{w}{z} \right) \right] \equiv wf_w \left( \frac{w}{z} \right) + zf_z \left( \frac{w}{z} \right) \equiv 0,$$

it follows that

$$\begin{aligned} f \left( \frac{w}{z} \right) L[F] &= f \left( \frac{w}{z} \right) F(w, z) + f \left( \frac{w}{z} \right) \omega[F(w, z)] = \\ &= f \left( \frac{w}{z} \right) F(w, z) + f \left( \frac{w}{z} \right) \omega[F(w, z)] + F(w, z) \omega \left[ f \left( \frac{w}{z} \right) \right] = \\ &= f \left( \frac{w}{z} \right) F(w, z) + \omega \left[ f \left( \frac{w}{z} \right) F(w, z) \right] = L \left[ f \left( \frac{w}{z} \right) F(w, z) \right]. \end{aligned}$$

Therefore, taking formula (1) into account,

$$\begin{aligned} &\frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{f(w/z)L^{(\Gamma)}[F]}{\zeta - u_0} d\zeta = \\ &= \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{L^{(\Gamma)}[f(w/z)F(w, z)]}{\zeta - u_0} d\zeta = f \left( \frac{w_0}{z_0} \right) F(w_0, z_0), \end{aligned}$$

whence the validity of formula (2) follows.

**Corollary 1.** If  $f(w/z) \equiv 1$ , then formula (2) becomes formula (1).

**Corollary 2.** If

$$F(w, z) = \frac{\varphi(w, z)}{\prod_{k=1}^m (w - \lambda_k z)}, \quad \lambda_k \quad (k = 1, 2, \dots, m)$$

are constants, then, putting

$$f(\xi) = \prod_{k=1}^m (\xi - \lambda_k),$$

we have

$$F(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} K(w, z; w_0, z_0) \frac{L_m^{(\Gamma)}[F]}{\zeta - u_0} d\zeta,$$

where  $K(w, z; w_0, z_0) = z^m f(w/z) : z_0^m f(w_0/z_0)$ ,  $L_m[F] = L[F] + mF(w, z)$ .

**Remark.** Formula (2) expresses the values of the function  $F(w, z)$  inside the domain  $D - P$ , where  $P$  is the manifold  $f(w/z) = 0$  belonging to  $D$ , in terms of the values of the linear differential operator  $L[F]$  on the part  $\Gamma$  of the boundary of the domain  $\overline{D}$ .

3. Let now  $f(w, z)$  be an arbitrary function regular in the domain  $D$ .

**Theorem 2.** *If the product  $f(w, z)F(w, z)$  is a regular function in the domain  $D$  and the operator  $L[fF]$  is continuous in the closed domain  $\overline{D}$ , then for  $(w_0, z_0) \in D$ , for  $f(w_0, z_0) \neq 0$ ,*

$$F(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} K(w, z; w_0, z_0) \frac{L_\psi^{(\Gamma)}[F]}{\zeta - u_0} d\zeta, \quad (3)$$

where

$$L_\psi[F] = L[F] + \psi(w, z)F, \quad \psi(w, z) = \omega(f)/f,$$

$$K(w, z; w_0, z_0) = f(w, z) : f(w_0, z_0), \quad (w, z) \in \Gamma.$$

**Proof.** Since

$$f(w, z)L_\psi[F] = f(w, z)L[F] + f(w, z)\psi(w, z)F(w, z) = f(w, z)L[F] + \omega(f)F = L[fF],$$

then, on the basis of formula (1),

$$\frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{f(w, z)L_\psi^{(\Gamma)}[F]}{\zeta - u_0} d\zeta =$$

$$= \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{L^{(\Gamma)}[fF]}{\zeta - u_0} d\zeta = f(w_0, z_0)F(w_0, z_0),$$

whence formula (3) follows. With respect to this theorem, the same remark holds as with respect to Theorem 1.

Let  $f_1(\xi)$  be an entire function and  $f_2(w, z)$  a regular function in the domain  $D$ .

A consequence of the two theorems indicated is

**Theorem 3.** If the product  $f_1(w/z)f_2(w, z)F(w, z)$  is a regular function in the domain  $D$  and the operator  $L_\psi[f_1(w/z)f_2(w, z)F(w, z)]$  is continuous in the closed domain  $\bar{D}$ , then for  $(w_0, z_0) \in D$ , but  $f_1(w_0/z_0)f_2(w_0, z_0) \neq 0$ ,

$$F(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\xi|=1} K(w, z; w_0, z_0) \frac{L_\psi^{(\Gamma)}[F]}{\xi - u_0} d\xi,$$

where  $L_\psi[F] = L[F] + \psi(w, z)F$ ,  $\psi(w, z) = \omega[f_2(w, z)]/f_2(w, z)$ ,  $K(w, z; w_0, z_0) = f_1(w/z)f_2(w, z) : f_1(w_0/z_0)f_2(w_0, z_0)$ ,  $(w, z) \in \Gamma$ .

4. Consideration of  $L_\psi[F] = wF'_w + zF'_z + [1 + \psi(w, z)]F$  has led to an integral representation of functions of the form  $F(w, z) = \varphi(w, z)/f_1(w/z)f_2(w, z)$ .

Let us now consider an operator of general form

$$L^*[F] = a(w, z)F'_w + b(w, z)F'_z + F(w, z).$$

It is easy to see that there exist independent functions  $\alpha(w, z)$  and  $\beta(w, z)$  which satisfy the system of equations:

$$\frac{\alpha\beta'_z - \beta\alpha'_z}{\Delta} = a(w, z), \quad \frac{\beta\alpha'_w - \alpha\beta'_w}{\Delta} = b(w, z),$$

where  $\Delta = \partial(\alpha, \beta)/\partial(w, z)$ . Indeed, these equations are equivalent to the following system of equations:

$$a(w, z) \frac{\partial\alpha}{\partial w} + b(w, z) \frac{\partial\alpha}{\partial z} = \alpha(w, z),$$

$$a(w, z) \frac{\partial\beta}{\partial w} + b(w, z) \frac{\partial\beta}{\partial z} = \beta(w, z).$$

Therefore, for the functions  $\alpha(w, z)$  and  $\beta(w, z)$  it is sufficient to take two particular independent solutions of the linear nonhomogeneous first-order partial differential equation

$$a(w, z) \frac{\partial v}{\partial w} + b(w, z) \frac{\partial v}{\partial z} = v(w, z).$$

This equation also predetermines the class of admissible functions  $a(w, z)$  and  $b(w, z)$ . Therefore

$$L^*[F] = \frac{\alpha\beta'_z - \beta\alpha'_z}{\Delta} F'_w + \frac{\beta\alpha'_w - \alpha\beta'_w}{\Delta} F'_z + F(w, z).$$

Let the equations

$$w = \gamma(x, y), \quad z = \delta(x, y)$$

define a generalized pseudoconformal mapping <sup>(4)</sup> of the domain  $D$  onto some internally branched domain  $D^*$ , and let  $x = \alpha(w, z)$ ,  $y = \beta(w, z)$  be the mapping of the domain  $D^*$  into  $D$ . Then

$$L^*[F] = xF'_x[\gamma(x, y), \delta(x, y)] + yF'_y[\gamma(x, y), \delta(x, y)] + F[\gamma(x, y), \delta(x, y)],$$

i.e.,

$$L^*[F(w, z)] = L[F(\gamma(x, y), \delta(x, y))].$$

Thus, consideration of the operator  $L^*[F(w, z)]$  leads to consideration of the linear differential operator  $L[\Phi(x, y)]$  studied by us, applied to the composite function  $\Phi(x, y) = F[\gamma(x, y), \delta(x, y)]$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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