

# ON LOCAL REGULARIZATIONS OF THE COEFFICIENT FUNCTIONS OF THE SCATTERING MATRIX

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**Abstract**

**Full Text**

**MATHEMATICAL PHYSICS**

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**ON LOCAL REGULARIZATIONS OF THE  
COEFFICIENT FUNCTIONS OF THE SCAT-  
TERING MATRIX**

*(Presented by Academician N. N. Bogolyubov on 30 XI 1961)*

When the scattering matrix  $S$  of quantum electrodynamics is expanded in a perturbation-theory series, as coefficient functions we obtain certain products of causal functions  $D^c(x)$  and  $S^c(x)$  <sup>(1)</sup>.

Since these functions are generalized functions and have singularities of the type  $(x^2 - i0)^{-1}$ ,  $\frac{\partial}{\partial x}(x^2 - i0)^{-1}$ , their products can no longer be directly regarded as continuous functionals on the Schwartz space  $S$ . This is expressed, in particular, in the fact that in computing the Fourier transforms of such products one has to deal with divergent integrals.

The essence of the methods proposed for eliminating such divergences consists, briefly, in regarding the corresponding product as a functional on some subspace of the Schwartz space, and then extending it in a definite way to the whole space. The infinities are thereby “subtracted” from the coefficient function. This “regularization” of products has in practice been carried out by smoothing the factors, followed by passage to the limit to the original functions. Such, for example, is the Pauli-Villars procedure <sup>(2-4)</sup>.

The method of regularization proposed below is also based on smoothing. But it is carried out only in a neighborhood of the vertex of the light cone, i.e., the value of the functions is not changed for values of their arguments whose modulus exceeds some arbitrarily small, prescribed number  $l > 0$ . It is in this sense that the proposed procedure is called local.

In what follows we shall proceed from direct products of causal functions of the form

$$K(x_1, \dots, x_n) = \prod_{k=1}^r D^c(x_{i_k} - x_{j_k}) \prod_{l=1}^s S_{\alpha_l \beta_l}^c(x_{i_l} - x_{j_l}). \quad (1)$$

It is obvious that, by multiplying the corresponding  $K(x)$  by constant matrices and taking convolutions over some pairs of indices, we can obtain any coefficient function of the matrix  $S$ .

The paper shows that to every  $K(x)$  one can assign a sequence of functions  $\overline{K}_\nu(x)$  having the following properties:

- 1)  $\overline{K}_\nu(x)$  decomposes into a product of a group of factors from (1), whose arguments satisfy  $|u_i| > l$ , and a certain generalized function of the remaining arguments; that is, in (1) the factors are smoothed only in the case when their arguments do not exceed  $l$ .
- 2) The sequence  $\overline{K}_\nu(x)$  has as its limit a certain generalized function on  $S$ . At the same time,  $\overline{K}_\nu(x)$  tends to  $K(x)$  at all points where the latter has meaning.
- 3) On that subspace of the Schwartz space on which the passage to the limit in the Pauli-Villars procedure is possible,  $\overline{K}_\nu(x)$  gives in the limit the same function as does the latter procedure.
- 4) Finally, we formulate the last property of the sequence  $\overline{K}_\nu(x)$ , which makes it possible to pass to nonlocal interactions. Namely, every function  $\overline{K}_\nu(x)$  can be represented in the form

$$\overline{K}_\nu(x) = \int \prod_{k=1}^r D^c(u_k - \xi_k) \prod_{j=1}^s S_{\alpha_j \beta_j}^c(v_j - \eta_j) g_{\beta_1 \gamma_1 \dots \beta_s \gamma_s}^\nu(\xi, \eta) d\xi d\eta, \quad (2)$$

where  $g^\nu(\xi, \eta)$  is some generalized matrix function that is different from zero only in a neighborhood of the origin of the coordinates, specified by the inequalities  $|\xi_k| < l$ ,  $|\eta_j| < l$  ( $k = 1, \dots, r$ ;  $j = 1, \dots, s$ ). We note that  $g^\nu(\xi, \eta)$  need not have as its limit some generalized function.

For the further discussion the following will be useful to us.

**Lemma.** If a generalized function  $\overline{S}^c(x)$  differs from  $S^c(x)$  only in the region  $|x| < l$ , then

$$\overline{S}^c(x) = \int S^c(x - \xi) g(\xi) d\xi,$$

where  $g(\xi) = 0$  for  $|\xi| > l$ .

Indeed,

$$(i\hat{\partial} - m)\overline{S}^c(x) = -g(x)$$

is a finite function. In terms of Fourier transforms this relation is written in the form

$$(\hat{k} + m)\widetilde{\overline{S}^c}(k) = \widetilde{g}(k).$$

Hence

$$\widetilde{\overline{S}^c}(k) = \frac{\hat{k} - m}{k^2 - m^2 + i0} \widetilde{g}(k),$$

since  $S^c(x) - \overline{S}^c(x)$  is a finite function. The expression obtained for  $\widetilde{\overline{S}^c}(k)$  proves our lemma.

An analogous fact can also be proved for  $D^c(x)$ .

A direct generalization of the lemma just proved is the following theorem, which is fundamental in our work.

**Theorem.** If the function  $\overline{K}_\nu(x)$  is obtained from some function

$$\overline{K}_\nu(u_1, \dots, u_r; v_1, \dots, v_s),$$

satisfying condition <sup>1</sup>, by replacing the variables  $u, v$  with the differences of the corresponding  $x$ 's, then  $\overline{K}_\nu(x)$  admits a representation of the form (2).

Thus, in order to prove the assertions stated above 1)–4), it is sufficient to construct a sequence  $\overline{K}_\nu(x)$  satisfying the condition of the theorem and conditions 2), 3).

Let us split each causal function, for example  $S^c(x)$ , into the sum of two components

$$S^c(x) = \sigma(x)S^c(x) + \omega(x)S^c(x).$$

Here  $\sigma(x)$  is an infinitely differentiable function, with  $\sigma(x) = 0$  for  $|x| > l$  and  $\sigma(x) = 1$  for  $|x| < l/2$ . The function  $\omega(x) = 1 - \sigma(x)$ .

The product (1) splits into a sum of products in which, in place of each factor, there will stand either the  $\sigma$ - or the  $\omega$ -component.

Consider one of these summands. If in it each  $\sigma$ -factor is replaced by a smoothed one (for example, by the Pauli-Villars method), then, as is not difficult to show, a generalized function on  $S$  is obtained.

Represent each such function in the form of a product

$$K_M^{(\sigma)}(x)K^{(\omega)}(x).$$

Here  $K_M^{(\sigma)}$  is the product of the  $\sigma$ -components smoothed by the Pauli-Villars method, while in  $K^{(\omega)}(x)$  all  $\omega$ -components are combined. It can be shown that for every such product there exists a counterterm of the form

$$\Pi\sigma(x)\Lambda_M(x)K^{(\omega)}(x),$$

such that the difference

$$[K_M^{(\sigma)}(x) - \Lambda_M(x)\Pi\sigma(x)]K^{(\omega)}(x)$$

is a linear continuous functional on  $S$  also after passage to the limit  $M \rightarrow \infty$ . Here  $\Lambda_M(x)$  depends only on the arguments of the  $\sigma$ -components.

This fact is a generalization of the corresponding result from the theory of the  $R$ -operation of N. N. Bogolyubov (see, for example, (3, 4)). Thus, we have shown that one can associate with the product (1) a sequence satisfying conditions 2),

3) and the conditions of the theorem. As was shown above, this completes the proof of assertions 1)–4).

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