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Abstract

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A SPECIAL CASE OF THE SOLUTION OF THE COAGULATION EQUATION

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The study of the process of planet formation by the aggregation of solid bodies and particles, whose existence may be assumed at a certain stage in the evolution of the circumsolar cloud, leads to consideration of the coagulation equation

$$\frac{\partial n(m,t)}{\partial t} = \frac{1}{2}\int_0^m A(m', m-m')\, n(m',t)\, n(m-m',t)\, dm' - n(m,t)\int_0^\infty A(m,m')\, n(m',t)\, dm', \tag{1}$$

where $n(m,t)$ is the number of bodies per unit volume and per unit mass interval m; $A(m,m')$ is the coagulation coefficient, depending on the masses of the colliding bodies m and m'. The classical solution of the coagulation equation for a constant value $A(m,m')$ and for a monodisperse initial state of the system was obtained by M. Smoluchowski ([1]). Subsequently, solutions were found for other initial distributions ([2–4]). In our problem the assumption that $A(m,m')$ is constant is unacceptable. For $A(m,m') \neq$ const, solving equation (1) presents considerable difficulties and in practice is usually carried out by approximate or numerical methods. However, these methods describe only the initial stage of the process and do not make it possible to trace the growth of bodies over enormous time intervals corresponding to an increase of mass by 8–10 orders of magnitude. It is therefore desirable to obtain an exact analytical solution of the problem, at least for some qualitatively plausible expression for the coagulation coefficient.

In the present article the coagulation coefficient is taken in the form

$$A(m,m') = A_1\,(m+m'), \tag{2}$$

where $A_1 =$ const, and a solution is found for the equation

$$\frac{\partial n(m,t)}{\partial t} = \frac{A_1}{2}\, m\int_0^m n(m',t)\, n(m-m',t)\, dm' - A_1 n(m,t)\int_0^\infty (m+m')\, n(m',t)\, dm'. \tag{1'}$$

In this equation, as in (1), it is assumed that the bodies only merge, and the possibility of their destruction in collisions is not taken into account. The collision frequency of small bodies is determined by their geometrical cross section, which is proportional to $m^{2/3}$. The effective cross section of large bodies, owing to their gravitational action, is much greater than the geometrical one and is proportional to $m^{4/3}$. Expression (2) for the coagulation coefficient is a kind of "mean" between these expressions for small and large bodies.

As noted earlier ([3,4]), for solving problems of this type it is expedient to use the integral Laplace transform. Let us introduce, instead of the desired distribution function $n(m,t)$, a new variable $g(m,t)$

$$n(m,t) = \varphi(t) e^{\chi(t)m} g(m,t), \tag{3}$$

where

$$\varphi(t) = e^{-A_1 M t}, \qquad \chi(t) = \frac{N_0}{M}\varphi(t), \tag{4}$$

$$N_0 = \int_0^\infty n(m,0)\,dm, \qquad M = \int_0^\infty m\,n(m,t)\,dm = \text{const}. \tag{5}$$

Then equation (1′) takes the form

$$\frac{1}{A_1\varphi(t)}\frac{\partial g}{\partial t} = \frac{m}{2}\int_0^m g(m-m',t)g(m',t)\,dm'. \tag{6}$$

Introduce a new variable τ

$$d\tau = M A_1 \varphi(t)\,dt, \qquad \tau = 1 - e^{-A_1 M t}. \tag{7}$$

As t varies from 0 to ∞, the quantity τ varies from 0 to 1. Applying the Laplace transform to equation (6),

$$G(p,\tau) = \int_0^\infty e^{-pm} g(m,\tau)\,dm \tag{8}$$

leads to a quasilinear partial differential equation for the function $G(p,\tau)$

$$M\frac{\partial G}{\partial \tau} + G\frac{\partial G}{\partial p} = 0. \tag{9}$$

The general solution of this equation has the form

$$G(p,\tau) = G_0(Mp - G(p,\tau)\tau), \tag{10}$$

where G_0 is an arbitrary function determined from the initial data. According to (8), (3), (4), and (10), for $\tau = t = 0$

$$G(p,0) = \int_0^\infty e^{-pm} g(m,0)\,dm = \int_0^\infty e^{-(p+N_0/M)m} n(m,0)\,dm = G_0(Mp).$$

Consequently,

$$G(p,\tau) = \int_0^\infty e^{-(Mp-\tau G(p,\tau)+N_0)m/M}\, n(m,0)\,dm. \tag{11}$$

Only in a few cases can $G(p,\tau)$ be found explicitly from this. If, for example, the initial distribution is taken in the form

$$n(m,0) = am^{-q}e^{-bm}, \tag{12}$$

then $G(p,\tau)$ is determined by the expression

$$G(p,\tau)\left(p + \frac{2-q}{1-q}b - \frac{\tau G(p,\tau)}{M}\right)^{1-q} = a\Gamma(1-q), \tag{13}$$

where $q < 1$. For $q = 1/2$ and -1, a cubic equation is obtained for $G(p,\tau)$, which can be solved. For other values of q, algebraic equations of higher degree are obtained for $G(p,\tau)$. We shall restrict ourselves to consideration of the simplest case $q = 0$. Then

$$n(m,0) = ae^{-bm}, \tag{14}$$

$$a = N_0^2/M = N_0/m_0, \qquad b = N_0/M = 1/m_0, \tag{15}$$

where m_0 is the average mass of a body at the initial instant. Substituting $n(m,0)$ from (14) into (11) and performing the integration, we find

$$G(p,\tau) = \frac{a}{p + 2b - M^{-1}\tau G(p,\tau)}, \tag{16}$$

whence

$$G(p,\tau)=\frac{M}{2\tau}\left[p+2b\pm\sqrt{(p+2b)^2-4b^2\tau}\right]. \tag{17}$$

A characteristic property of the quasilinear equation (9) is the nonuniqueness of its solutions[5]. In solution (17) this manifests itself in the two signs before the root. The branch point is located at $p=-2b(1-\sqrt{\tau})$. It moves from $p=-2b$ at $\tau=0$ to $p=0$ at $\tau=1$, i.e., at $t=\infty$. A definite single-valued solution can be obtained only if the branch point lies at all times outside the range of values of p under consideration. On the other hand, the function $G(p,\tau)$, obtained by the Laplace transform, is defined in the complex half-plane $\mathrm{Re}\,p>s_0$, where s_0 is the growth exponent of the original $g(m,\tau)$. From the solution (19) for $g(m,\tau)$ found below, it is clear that $s_0=-2b$ when $\tau=0$, and $s_0=0$ when $\tau=1$. Thus, both restrictions on p coincide at the endpoints of the interval of variation of τ. Taking $p>0$, we shall for all values of τ simultaneously satisfy both the condition imposed on the original function and the condition of absence of branch points in the considered domain of variation of the variables. The latter turns out to be possible only because in equation (9) the variable τ varies over a bounded interval: with unbounded increase of time, $\tau\to 1$*.

Of the two branches of solution (17), only one is suitable—the one with the negative sign before the root, since only in this case does $G(p,\tau)$ remain bounded as $\tau\to 0$ and tend to zero as $p\to\infty$, as it should according to (8). Consequently, the desired solution of equation (9) for the initial distribution (14) must have the form

$$G(p,\tau)=\frac{M}{2\tau}\left[p+2b-\sqrt{(p+2b)^2-4b^2\tau}\right]. \tag{18}$$

To pass from the image $G(p,\tau)$ to the original function $g(m,\tau)$ (the inverse Laplace transform), we use formulas from the handbook[6]. After some transformations we find

$$g(m,\tau)=\frac{N_0}{m\sqrt{\tau}}e^{-2bm}I_1\left(2bm\sqrt{\tau}\right). \tag{19}$$

Finally, with the aid of (3), taking (15) into account, we pass to the original distribution function $n(m,\tau)$

$$n(m,\tau)=\frac{N_0(1-\tau)}{m\sqrt{\tau}}e^{-bm(1+\tau)}I_1\left(2bm\sqrt{\tau}\right). \tag{20}$$

The modified Bessel function $I_1(x)$ has the following expansions: for $x\ll 1$

$$I_1(x)=\frac{x}{2}\left[1+\frac{(x/2)^2}{1^2\cdot 2}+\frac{(x/2)^4}{1^2\cdot 2^2\cdot 3}+\ldots\right], \tag{21}$$

for $x \gg 1$

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{3}{8x} - \frac{15}{128x^2} - \dots\right). \tag{22}$$

* It should be noted that an analogous result is also obtained for other initial distributions $n(m, 0)$. It is easy, for example, to show that for $n(m, 0)$ taken in the form (12) or as a δ-function, in the real p-axis all the singular points lie in the region of negative values of p for all $\tau < 1$. In the case of complex p, for sufficiently large $\mathrm{Re}\, p$ the real and imaginary parts of $G(p, \tau)$, as is evident from (11), are small on the correctly chosen branch. But for a small imaginary part, the solution for $G(p, \tau)$ differs little from the solution for real p. Consequently, also for complex p the lower boundary $\mathrm{Re}\, p$ can be chosen so that the branch points of $G(p, \tau)$ lie at all times outside the limits of the region under consideration.

Accordingly, the distribution function of bodies by their masses (20) can be approximated as follows:

for $2m\sqrt{\tau} \ll m_0$

$$n(m, \tau)\, dm \simeq N_0(1 - \tau) e^{-(1+\tau)m/m_0} \frac{dm}{m_0} \tag{23}$$

and for $2m\sqrt{\tau} \gg m_0$

$$n(m, \tau)\, dm \simeq \frac{N_0(1 - \tau)}{2\sqrt{\pi}\tau^{3/4}} \left(\frac{m}{m_0}\right)^{-3/2} e^{-(1-\sqrt{\tau})^2 m/m_0} \frac{dm}{m_0}. \tag{24}$$

Only for the early stage of the process of coalescence and for small values of m is the condition $2m\sqrt{\tau} \ll m_0$ satisfied, and the distribution function is exponential. For the greater part of the range of values of m and τ, one may use expression (24), which is the product of a power function of m and an exponential. For large t, the value of τ is close to 1, and $(1 - \sqrt{\tau})^2$ is very small. Therefore the exponential function in this case begins to play an essential role only for the very largest values of m. Almost over the entire interval of variation of m (except for the very largest and the very smallest m), $n(m) \propto m^{-3/2}$ holds. We note that this power function is close to the mass distributions of small bodies of the solar system found from observations—comets, asteroids, and meteorites falling to the Earth. The exponent $-3/2$ of m does not depend on the parameters a and b of the initial distribution. One may suppose that it depends only on the form of the coagulation coefficient $A(m, m')$ and has the same numerical value for other initial distributions, even those differing substantially from (14).

In the distribution (24), a significant part of the mass of the system falls to the share of large bodies; moreover, with the passage of time the relative mass of large bodies increases. At the final stage of the Earth' s growth, according to (24), half of the mass was supplied by bodies larger than the Moon. This may be interpreted to mean that the greater part of the material that formed the Earth consisted of large bodies. Differences in the chemical composition of the bodies could therefore have created noticeable inhomogeneities inside the Earth. However, such a conclusion can be regarded as fully correct only after considering a more complicated coagulation problem that takes into account the fragmentation of colliding bodies. Fragmentation increases the mass of small bodies, and taking it into account may noticeably alter the quantitative results. Nevertheless, the qualitative conclusion about the significant role of large bodies will apparently remain valid.

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Note: Figure translations are in progress. See original paper for figures.