ON THE RITZ METHOD IN NONLINEAR PROBLEMS

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Abstract

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S. G. MIKHLIN

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In a number of works the existence of a solution of a variational problem for functionals of a fairly general form is established by proving the convergence of a minimizing sequence; among recent works on this topic we note the papers ([1,2]). However, the important question of the actual construction of a minimizing sequence in the general case has remained open. For quadratic functionals bounded below, such a construction can be carried out by applying the Ritz method ([3]); I. I. Vorovich showed ([4]) that the Ritz method gives a minimizing sequence (whose convergence was proved by the same author) for functionals of the nonlinear theory of shells; an analogous result was obtained by A. Langenbach for some functionals of the theory of plasticity in the papers ([2]).

The purpose of the present article is to prove the following theorem.

Theorem. Let the functional $\Phi(u)$ satisfy the following requirements: a) its domain of definition $D(\Phi)$ is a linear set; b) on every finite-dimensional linear manifold in its domain of definition this functional is continuously differentiable; c) in some metric ρ, defined on $D(\Phi)$, the functional in question is increasing and lower semicontinuous.

Let, further, the coordinate sequence $\{\varphi_n\}$ be subject to the usual conditions: 1) $\varphi_n \in D(\Phi)$; 2) the elements $\varphi_1, \varphi_2, \dots, \varphi_n$ are linearly independent for every n; 3) the coordinate sequence is complete in the metric ρ.

Under the enumerated conditions, the Ritz method gives a minimizing sequence for the functional $\Phi(u)$.

Proof. For any n one can construct the n-th approximate solution by Ritz' s method. Indeed, the expression $\Phi\left(\sum_{k=1}^{n} a_k\varphi_k\right)$ is a function of the variables $a_1, a_2, \dots, a_n$, continuously differentiable for all values of these variables and tending to $+\infty$ if $a_1^2 + a_2^2 + \dots + a_n^2 \to \infty$. This function attains its absolute minimum at least at one point, which is at a finite distance from the origin, and at this point

$$\frac{\partial\Phi\left(\sum_{k=1}^{n} a_k\varphi_k\right)}{\partial a_j} = 0, \qquad j = 1, 2, \dots, n.$$

We shall now prove that the approximate Ritz solutions form a minimizing sequence for the functional $\Phi(u)$. Let $\inf \Phi(u) = d$. Construct a minimizing sequence $\{u^{(n)}\}$ such that

$$\Phi\left(u^{(n)}\right) \leq d + \frac{1}{n}.$$

In view of condition 3), for each $u^{(n)}$ one can choose such a linear combination

$$v^{(N_n)} = \sum_{k=1}^{N_n} \alpha_k^{(n)} \varphi_k,$$

that $\rho(u^{(n)}, v^{(N_n)}) < \delta_n$; choose the number δ_n so small that, for any v satisfying the inequality $\rho(u^{(n)}, v) < \delta_n$, one has $\Phi(u^{(n)}) - \Phi(v) \geqslant -\frac{1}{n}$. Then

$$\Phi(v^{(N_n)}) \leqslant \Phi(u^{(n)}) + \frac{1}{n}$$

and, a fortiori,

$$\Phi(v^{(N_n)}) \leqslant d + \frac{2}{n}.$$

It follows that $\{v^{(N_n)}\}$ is a minimizing sequence. Let

$$u_p = \sum_{k=1}^{p} a_k \varphi_k$$

denote the p-th approximate solution by Ritz' s method. Then

$$\Phi(u_{N_n}) \leqslant \Phi(v^{(N_n)}) \leqslant d + \frac{2}{n},$$

and $\{\Phi(u_{N_n})\}$ is also a minimizing sequence. Finally, since $\Phi(u_n)$ decreases monotonically as n increases, while the sequence $\Phi(u_{N_n}) \to d$, it follows that $\Phi(u_n) \to d$, as was required to prove.

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Note: Figure translations are in progress. See original paper for figures.