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Abstract

Full Text

GEOPHYSICS

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INTEGRATION OF THE TIDAL EQUATIONS IN ONE CASE OF UNSTEADY MOTION

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Let us consider the propagation of unsteady long waves in a basin extending infinitely in the horizontal directions and rotating with constant angular velocity ω about a vertical axis, of variable depth, which for $y > c$ is equal to h_1, for $0 < y < c$ is equal to h, and for $y < 0$ is equal to h_2. It is assumed that in the strip $b - \varepsilon < y < b + \varepsilon$, at the initial instant $t = 0$, an elevation of the fluid $\zeta = \text{const}$ is formed, and the subsequent propagation of the initial elevation of the fluid is studied within the assumptions of the theory of long waves. The problem is reduced to the integration of the equations of long waves

$$\begin{gathered} \frac{\partial u_j}{\partial t} - 2\omega v_j = 0, \\ \frac{\partial v_j}{\partial t} + 2\omega u_j = -g\frac{\partial \zeta_j}{\partial y}, \\ \frac{\partial \zeta_j}{\partial t} = -h_j\frac{\partial v_j}{\partial y}. \end{gathered} \tag{1}$$

By the index 1 we denote quantities referring to the part of the basin of depth h_1, by the index 2—to the part of the basin of depth h_2; without an index are written the quantities referring to the part of the basin of depth h.

The boundary conditions consist in the equality of elevations and discharges at the places where the depth changes; they have the form

$$\zeta_1 = \zeta, \quad h_1 v_1 = hv \qquad \text{for } y = c < b - \varepsilon; \tag{2}$$

$$\zeta = \zeta_2, \quad hv = h_2 v_2 \qquad \text{for } y = 0. \tag{3}$$

The initial conditions are prescribed in the form

$$\begin{gathered}\zeta_1(y,0)=\zeta_0=\text{const}\quad\text{for } t=0 \text{ and } b-\varepsilon<y<b+\varepsilon;\\ \zeta_1(y,0)=0\quad\text{for } t=0 \text{ and } y>b+\varepsilon \text{ or } y<b-\varepsilon;\\ u_1(y,0)=v_1(y,0)=0\quad\text{for } t=0.\end{gathered}\tag{4}$$

The solution of systems (1) under the boundary conditions (2) and (3) and the initial conditions (4) is sought by the operational method.

Omitting the intermediate calculations in this brief note, we give the integral expression, for example, for the elevation of the fluid in the region $y<0$:

$$\zeta_2=\frac{\zeta_0}{\pi i}\frac{\sqrt{hh_1}}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})}\times$$

$$\times\left\{\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp\left[-\sqrt{s^2+4\omega^2}\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}}+\frac{c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}\right)\right]\exp(st)}{1-H\exp\left[-\dfrac{2c\sqrt{s^2+4\omega^2}}{\sqrt{gh}}\right]}\frac{ds}{s}-\right.$$

$$\left.-\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp\left[-\sqrt{s^2+4\omega^2}\left(\frac{b+\varepsilon-c}{\sqrt{gh_1}}+\frac{c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}\right)\right]\exp(st)}{1-H\exp\left[-\dfrac{2c\sqrt{s^2+4\omega^2}}{\sqrt{gh}}\right]}\frac{ds}{s}\right\}.\tag{5}$$

where

$$H=\frac{(\sqrt{h}-\sqrt{h_1})(\sqrt{h}-\sqrt{h_2})}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})}.\tag{6}$$

For the unique determination of the integrals entering into this expression, we draw cuts from the branch points of the root $\sqrt{s^2+4\omega^2}$ along the imaginary axis, and assume that on the first sheet, over which the integration is performed, the branch is chosen for which, for real values of s, the root takes positive values. Then the integrand in expression (5) can be expanded in a convergent series, and consequently the elevation ζ_2 can be represented in the form of integrals of a convergent series

$$\begin{aligned}\zeta_2 = \frac{\zeta_0}{\pi i}\,\frac{\sqrt{hh_1}}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})}\times\\ \times\int_{\sigma-i\infty}^{\sigma+i\infty}\Bigg\{\exp\left[-\sqrt{s^2+4\omega^2}\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}}+\frac{c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}\right)\right]\times\\ \times\exp(st)\left[1+\sum_{n=1}^{\infty}H^n\exp\left(-\frac{2nc\sqrt{s^2+4\omega^2}}{\sqrt{gh}}\right)\right]-\\ -\exp\left[-\sqrt{s^2+4\omega^2}\left(\frac{b+\varepsilon-c}{\sqrt{gh_1}}+\frac{c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}\right)\right]\times\\ \times\exp(st)\left[1+\sum_{n=1}^{\infty}H^n\exp\left(-\frac{2nc\sqrt{s^2+4\omega^2}}{\sqrt{gh}}\right)\right]\Bigg\}\frac{ds}{s}.\end{aligned}\tag{7}$$

It is shown that for $t < Y$ an integral of the form

$$I = \int_{\sigma-i\infty}^{\sigma+i\infty}\exp\left[-Y\sqrt{s^2+4\omega^2}\right]\frac{\exp(st)}{s}\,ds \tag{8}$$

vanishes, while for $t > Y$

$$I = 2\pi i\exp(-2\omega Y) - 4i\int_1^{\infty}\sin\left(2\omega Y\sqrt{\xi^2-1}\right)\cos(2\omega t\xi)\frac{d\xi}{\xi}. \tag{9}$$

Using this result, we obtain that the elevation ζ_2 in the part of the basin $y < 0$, up to the time

$$t_1 = \frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}},$$

remains equal to zero.

To obtain the values of ζ_2 with a further increase of time t, we represent ζ_2 in the form of a sum

$$\zeta_2 = \zeta_2' + \zeta_2''; \tag{10}$$

then, for

$$\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}} < t < \frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{3c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}},$$

we shall have the elevation

$$\zeta_2' = 2\zeta_0 \frac{\sqrt{hh_1}}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})} \left\{ \exp\left[-2\omega\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right] - \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b-}{\sqrt{}}\right.\right.\right. \tag{11}$$

for

$$\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{3c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}} < t < \frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{5c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}.$$

elevation

$$\begin{aligned}\zeta_2' = 2\zeta_0 \frac{\sqrt{hh_1}}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})} &\left\{ \exp\left[-2\omega\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right]\right. \\ &- \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right]\cos 2\omega t\xi\,\frac{d\xi}{\xi} \\ &+ H\left[\exp\left[-2\omega\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{3c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right]\right. \\ &\quad\left.\left. - \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{3c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right]\cos 2\omega t\xi\,\frac{d\xi}{\xi}\right]\right\}.\end{aligned} \tag{12}$$

Comparison of formulas (11) and (12) shows that, with an increase in time by $2c/\sqrt{gh}$, two new terms are added to the result, and thus, for

$$\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{(2N-1)c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}} < t < \frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{(2N+1)c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}$$

elevation

$$\begin{aligned}\zeta_2' = 2\zeta_0 &\frac{\sqrt{hh_1}}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})} \times \\ &\times \sum_{n=1}^{N} H^{n-1}\left\{\exp\left[-2\omega\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{(2n-1)c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right] - \right. \\ &\quad\left. - \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b-\varepsilon-c}{\sqrt{gh_1}} + \frac{(2n-1)c}{\sqrt{gh}} - \frac{y}{\sqrt{gh_2}}\right)\right]\cos 2\omega t\xi\,\frac{d\xi}{\xi}\right\}.\end{aligned} \tag{13}$$

An analogous equality may be written for ζ_2'', with the difference that everywhere the quantity $-\varepsilon$ must be replaced by $+\varepsilon$, and the sign before the entire expression must be changed to the opposite. We have, for

$$\frac{b+\varepsilon-c}{\sqrt{gh_1}}+\frac{(2N-1)c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}<t<\frac{b+\varepsilon-c}{\sqrt{gh_1}}+\frac{(2N+1)c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}$$

elevation

$$\begin{aligned}\zeta_2'' = & -2\zeta_0\frac{\sqrt{hh_1}}{(\sqrt{h}+\sqrt{h_1})(\sqrt{h}+\sqrt{h_2})}\times\\ & \times\sum_{n=1}^{N}H^{n-1}\left\{\exp\left[-2\omega\left(\frac{b+\varepsilon-c}{\sqrt{gh_1}}+\frac{(2n-1)c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}\right)\right]-\right.\\ & \left.-\frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b+\varepsilon-c}{\sqrt{gh_1}}+\frac{(2n-1)c}{\sqrt{gh}}-\frac{y}{\sqrt{gh_2}}\right)\right]\cos 2\omega t\xi\,\frac{d\xi}{\xi}\right\}.\end{aligned} \tag{14}$$

Formulas (10), (13), and (14) give a clear picture of the propagation of disturbances over the surface of the liquid. After the moment of time $t = t_1$, corresponding to the arrival at the observation point of the front edge of the direct wave, the elevation is determined by formulas (13) and (14). Each newly appearing term corresponds to the arrival of the front or rear edge of disturbances successively reflected from the boundaries of the ledge. Since all the reflection coefficients are quantities less than unity, the elevation determined by each subsequent reflection has a smaller magnitude.

The decrease of the elevation after each subsequent reflection is also determined by the increase of the exponent in the exponential factor. Analogous formulas are obtained for the elevation of the liquid in the parts of the basin of depth h_1 and depth h.

Let us consider the particular case in which there is only one ledge in the basin and the depth of the basin is equal to h_1 for $y > 0$ and to h_2 for $y < 0$. In this case, for

$$t<\frac{b-\varepsilon}{\sqrt{gh_1}}-\frac{y}{\sqrt{gh_2}}$$

the elevation is

$$\zeta_2 = 0;$$

for

$$\frac{b-\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}} < t < \frac{b+\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}$$

the elevation is

$$\zeta_2 = \zeta_0 \frac{\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} \left\{ \exp\left[-2\omega\left(\frac{b-\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}\right)\right] - \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b-\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}\right)\right] \cos 2\omega t\xi \right. \tag{15}$$

for

$$t > \frac{b+\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}$$

the elevation is

$$\begin{aligned} \zeta_2 = \zeta_0 \frac{\sqrt{h_1}}{\sqrt{h_1}+\sqrt{h_2}} \Bigg\{ & \exp\left[-2\omega\left(\frac{b-\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}\right)\right] \\ & - \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b-\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}\right)\right] \cos 2\omega t\xi \,\frac{d\xi}{\xi} \\ & - \exp\left[-2\omega\left(\frac{b+\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}\right)\right] \\ & + \frac{2}{\pi}\int_1^\infty \sin\left[2\omega\sqrt{\xi^2-1}\left(\frac{b+\varepsilon}{\sqrt{gh_1}} - \frac{y}{\sqrt{gh_2}}\right)\right] \cos 2\omega t\xi \,\frac{d\xi}{\xi} \Bigg\}. \end{aligned} \tag{16}$$

The integrals occurring in formulas (13)–(16) are investigated asymptotically for large values of time for a moving and a stationary observer.

Carrying out, for example, an asymptotic analysis of integrals of the form

$$I_1 = \frac{2}{\pi}\int_1^\infty \sin\left(2\omega Y\sqrt{\xi^2-1}\right)\cos 2\omega t\xi \,\frac{d\xi}{\xi}$$

for large values of time and constant Y, we obtain

$$I_1 \simeq -\frac{Y}{2\sqrt{\pi\omega}\,t}\sin\left(2\omega t + \frac{\pi}{4}\right).$$

Thus, at each point the part of the elevation characterized by an integral of the type I_1 tends to zero with increasing time as $t^{-3/2}$. All these integrals can also be calculated by numerical methods for each particular case.

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Note: Figure translations are in progress. See original paper for figures.