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Abstract

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THE METHOD OF FRACTIONAL STEPS FOR SOLVING THE CAUCHY PROBLEM FOR A MULTIDIMENSIONAL WAVE EQUATION

(Presented by Academician S. L. Sobolev, 26 V 1962)

The method of fractional steps, initially created for the effective solution of the multidimensional heat-conduction equation, has recently also been successfully applied to the solution of other problems. We note, in particular, the works ([1,2]). In the present note, difference schemes (8), (12) for solving the Cauchy problem for a multidimensional wave equation are proposed and investigated; these schemes are also based on the method of fractional steps.

Consider the equation:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}, \tag{1}$$

for which we pose the periodic Cauchy problem:

$$u(0, x_1, \dots, x_N) = \varphi(x_1, \dots, x_N),$$

$$\frac{\partial u}{\partial t}(0, x_1, \dots, x_N) = \psi(x_1, \dots, x_N), \tag{2}$$

$$-\infty < x_1, \dots, x_N < \infty, \qquad t > 0.$$

Let

$$T_{\pm i} f(x_1, \dots, x_N) = f(x_1, \dots, x_i \pm h_i, \dots, x_N),$$

$$\Lambda_{ii} = \frac{1}{h_i^2}(T_i - 2E + T_{-i}), \tag{3}$$

$$u^n(x_1, \dots, x_N) = u(n\tau, x_1, \dots, x_N), \qquad \tau > 0.$$

We choose the following difference approximation to problem (1)–(2):

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} = \sum_{i=1}^{N} \Lambda_{ii} u^{n+1},$$

$$u^0(x_1, \dots, x_N) = \varphi(x_1, \dots, x_N), \tag{4}$$

$$u^1(x_1, \dots, x_N) = \varphi(x_1, \dots, x_N) + \tau\psi(x_1, \dots, x_N).$$

The method of fractional steps for solving problem (4) can be carried out by means of various difference schemes. We shall first dwell on the so-called splitting scheme, proposed in ([3]) for the multidimensional heat-conduction equation.

Following (4), we write the difference equation from (4) in the form

$$\left(E - \tau^2 \sum_{i=1}^{N} \Lambda_{ii}\right) u^{n+1} = 2u^n - u^{n-1}. \tag{5}$$

We have

$$\left(E - \tau^2 \sum_{i=1}^{N} \Lambda_{ii}\right) u^{n+1} = \prod_{i=1}^{N} (E - \tau^2 \Lambda_{ii}) u^{n+1} + O(\tau^4).$$

We now replace difference equation (5) by the factorized equation:

$$\prod_{i=1}^{N} (E - \tau^2 \Lambda_{ii}) u^{n+1} = 2u^n - u^{n-1}. \tag{6}$$

Assuming

$$(E - \tau^2 \Lambda_{11}) u^{n+\frac{1}{N}} = 2u^n - u^{n-1},$$

$$\dots\dots\dots\dots\dots\dots$$

$$(E - \tau^2 \Lambda_{pp}) u^{n+\frac{p}{N}} = u^{n+\frac{p-1}{N}},$$

$$\dots\dots\dots\dots\dots \tag{7}$$

$$(E-\tau^2\Lambda_{NN})u^{n+1}=u^{n+\frac{N+1}{N}},$$

we arrive at the splitting scheme:

$$\frac{u^{n+\frac{1}{N}}-2u^n+u^{n-1}}{\tau^2}=\Lambda_{11}u^{n+\frac{1}{N}},$$

$$\frac{u^{n+\frac{p}{N}}-u^{n+\frac{p-1}{N}}}{\tau^2}=\Lambda_{pp}u^{n+\frac{p}{N}},\qquad p=2,\dots,N, \tag{8}$$

$$u^0(x_1,\dots,x_N)=\varphi(x_1,\dots,x_N),$$

$$u^1(x_1,\dots,x_N)=\varphi(x_1,\dots,x_N)+\tau\psi(x_1,\dots,x_N)$$

for the solution of problem (1)–(2). Eliminating from here the auxiliary quantities $u^{n+\frac{p}{N}}$ $(p=1,\dots,N-1)$, we obtain

$$\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}=\sum_{i=1}^{N}\Lambda_{ii}u^{n+1}+\sum_{m=2}^{N}(-1)^{m+1}\tau^{2\bar{m}}D_{mu}^{n+1}, \tag{9}$$

where

$$D_m=\sum_{i_1\cdots i_m}\Lambda_{i_1i_1}\cdots\Lambda_{i_mi_m},\qquad \bar{m}=m-1 \tag{10}$$

and the summation in (10) is over combinations of indices $(i_1,\dots,i_m)$, which run through the first m values from $1,2,\dots,N$.

Using (9), we conclude that the difference scheme (8) approximates problem (1)–(2), and this approximation is consistent under the limiting transition law $\frac{\tau}{h_i}=\text{const}$ $(i=1,2,\dots,N)$. Let $b_i=2\frac{\tau}{h_i}\sin\frac{k_ih_i}{2}$. Application of Fourier analysis to scheme (8) shows that the ρ-factor of transition from the n-th to the $(n+1)$-st step satisfies the quadratic equation:

$$\rho^2-\frac{2\rho}{A_N}+\frac{1}{A_N}=0,\qquad A_N=(1+b_1^2)\dots(1+b_N^2), \tag{11}$$

the roots of which always lie inside the unit circle, independently of the value of the ratio $\tau/h_i=\text{const}$. Therefore the convergence of the solution of the difference scheme (8) to the solution of problem (1)–(2) follows from the equivalence theorem (5).

Along with scheme (8), the method of fractional steps for solving problem (1)–(2) can be realized by means of the following difference scheme:

$$\frac{u^{n+\frac{1}{N}} - 2u^n + u^{n-1}}{\tau^2} = \Lambda_{11} u^{n+\frac{1}{N}} + \sum_{i=2}^{N} \Lambda_{ii} u^n,$$

$$\frac{u^{n+\frac{p}{N}} - u^{n+\frac{p-1}{N}}}{\tau^2} = \Lambda_{pp}\left(u^{n+\frac{p}{N}} - u^n\right), \qquad p = 2, \dots, N, \tag{12}$$

$$u^0(x_1, \dots, x_N) = \varphi(x_1, \dots, x_N),$$

$$u^1(x_1, \dots, x_N) = \varphi(x_1, \dots, x_N) + \tau\psi(x_1, \dots, x_N).$$

Scheme (12) is an analogue of the difference scheme proposed in (6) for the multidimensional heat equation. The scheme equivalent to (12)

without auxiliary quantities $u^{n+\frac{p}{N}}$ $(p = 1, \dots, N-1)$ has the form:

$$\begin{gathered}\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} = \sum_{i=1}^{N} \Lambda_{ii} u^{n+1} + \sum_{m=2}^{N} (-1)^{m+1} \tau^{2m} D_m\left(u^{n+1} - u^n\right), \\ u^0(x_1, \dots, x_N) = \varphi(x_1, \dots, x_N), \\ u^1(x_1, \dots, x_N) = \varphi(x_1, \dots, x_N) + \tau\psi(x_1, \dots, x_N),\end{gathered} \tag{13}$$

whence it follows that the approximation of problem (1)–(2) by the difference scheme (12) is consistent. The transition multiplier of the difference scheme (12) satisfies the equation

$$\rho^2 - \frac{1 + A_N - (b_1^2 + \dots + b_N^2)}{A_N}\rho + \frac{1}{A_N} = 0, \tag{14}$$

for whose roots, independently of the magnitude of the ratio $\tau/h_i = \text{const}$, the condition $|\rho_{1-2}| \leqslant 1$ is fulfilled. On the basis of the equivalence theorem we conclude that scheme (12) is convergent.

Restricting ourselves to the case $N = 2$, we note that the difference scheme (12) is applicable also to the more general equation:

$$\frac{\partial^2 u}{\partial t^2} = a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2}, \tag{15}$$

where $a_{ij} = \text{const}$ $(i, j = 1, 2)$, $a_{11} > 0$, $a_{11}a_{22} - a_{12}^2 > 0$.

Let

$$\Lambda_{12} = \frac{1}{4h_1 h_2}(T_1 T_2 - T_1 T_{-2} - T_{-1} T_2 + T_{-1} T_{-2}).$$

Then the difference scheme (12) for solving problem (15)–(2) has the following form:

$$\begin{gathered} \frac{u^{n+1/2} - 2u^n + u^{n-1}}{\tau^2} = a_{11}\Lambda_{11} u^{n+1/2} + 2a_{12}\Lambda_{12} u^n + a_{22}\Lambda_{22} u^n, \\ \frac{u^{n+1} - u^{n+1/2}}{\tau^2} = a_{22}\Lambda_{22}\left(u^{n+1} - u^n\right), \\ u^0(x_1, x_2) = \varphi(x_1, x_2), \qquad u^1(x_1, x_2) = \varphi(x_1, x_2) + \tau\psi(x_1, x_2). \end{gathered} \tag{16}$$

Scheme (16) approximates problem (15)–(2), and the conditions $a_{11} > 0$, $a_{11}a_{12} - a_{12}^2 > 0$ are sufficient for spectral stability. Consequently, the difference scheme (16) is convergent.

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