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Abstract

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MATHEMATICS

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ON INTEGRAL REPRESENTATIONS OF FINITE GROUPS

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Higman ([1]) showed that a finite group G and its Sylow p-subgroup simultaneously have a finite or an infinite number of indecomposable representations over an arbitrary field K of characteristic p. The same result holds for representations of finite groups over the ring of integral p-adic numbers I_p. Since an abelian p-group of type (p, p), according to ([2]), has over the ring I_p indecomposable representations of arbitrarily high degree, the question of finite groups with a finite number of indecomposable representations over this ring reduces to the analogous question for cyclic primary (with respect to p) groups. Diederichsen ([3]) showed that a group of order p has over the ring I_p and over the ring of rational integers Z a finite number of indecomposable representations, and the degrees of these representations do not exceed the order of the group. A. V. Roiter ([4]) proved the same result for representations of the cyclic group of order 4 over the rings I_p and Z. (The series of indecomposable representations of arbitrarily large degree given in ([3]) turned out to be erroneous.)

In the present note the question of finite groups admitting only a finite number of indecomposable representations over the ring I_p is completely solved, and a number of theorems on representations of finite groups over the ring of rational integers Z are given.

Let $G = (a)$ be a cyclic group of order p^2 (p prime). Denote by ξ and ε, respectively, primitive roots of degree p^2 and p of 1. Form the rings $I_p[\xi]$ and $I_p[\varepsilon]$ and choose in them I_p-bases:

$$1,\ \xi,\ \dots,\ \xi^{\varphi(p^2)-1}; \tag{1}$$

$$1,\ \varepsilon,\ \dots,\ \varepsilon^{\varphi(p)-1}. \tag{2}$$

Let $\tilde{\xi}$ (respectively $\tilde{\varepsilon}$) be the matrix corresponding to the operator of multiplication by ξ (respectively by ε) in the basis (1) (respectively (2)). The group G has 3 irreducible representations over the ring I_p: $a \to 1$; $a \to \tilde{\varepsilon}$; $a \to \tilde{\xi}$.

If a representation

$$a \to \begin{pmatrix} \tilde{\xi} & A \\ 0 & \tilde{\varepsilon} \end{pmatrix}$$

is given, then to each column

$$\begin{pmatrix} a_{1j} \\ \vdots \\ a_{kj} \end{pmatrix}$$

($k = \varphi(p^2)$; φ is Euler' s function) one may put in correspondence the element $a_{1j} + \dots + a_{kj}\xi^{k-1}$ of the ring $I_p[\xi]$.

Theorem 1. *All indecomposable representations of the group $G = (a)$ $(a^{p^2} = 1;\ p > 2)$ over the ring I_p, up to equivalence, are exhausted by the following series:*

$$1)\ 1; \qquad 2)\ \tilde{\varepsilon}; \qquad 3)\ \tilde{\xi}; \qquad 4)\ \begin{pmatrix} \tilde{\xi} & (t) \\ 0 & \tilde{\varepsilon} \end{pmatrix}; \qquad 5)\ \begin{pmatrix} \tilde{\xi} & (1) \\ 0 & 1 \end{pmatrix}; \qquad 6)\ \begin{pmatrix} \tilde{\varepsilon} & (1) \\ 0 & 1 \end{pmatrix};$$

$$7)\ \begin{pmatrix} \tilde{\xi} & (t) & 0 \\ 0 & \tilde{\varepsilon} & (1) \\ 0 & 0 & 1 \end{pmatrix}; \qquad 8)\ \begin{pmatrix} \tilde{\xi} & 0 & (1) \\ 0 & \tilde{\varepsilon} & (1) \\ 0 & 0 & 0 \end{pmatrix}; \qquad 9)\ \begin{pmatrix} \tilde{\xi} & (t) & (1) \\ 0 & \tilde{\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t = 1, \dots, \varphi(p) - 1;$$

$$10)\ \Gamma_t(1) = \begin{pmatrix} \tilde{\xi} & (t) & 0 & (1) \\ 0 & \tilde{\varepsilon} & (1) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (t = 2, \dots, \varphi(p) - 1).$$

Here (t) denotes a rectangular matrix with zero columns except for the last one, whose last column corresponds to the element $(1-\xi)^{t-1} \in I_p[\xi]^*$.

For $p = 2$ the first 9 series of Theorem 1 give the indecomposable Z-representations of the cyclic group of order 4, indicated by A. V. Roiter ([4]). The degrees of the representations of series 1)–9) do not exceed the order of the group, while series 10) gives representations of degree $p^2 + 1$ (it appears only for $p > 2$).

Theorem 2. *The cyclic group $H = (a)$ of order p^3 (p an arbitrary prime) has, over the ring I_p, indecomposable representations of arbitrarily large degree that can be realized by integer matrices.*

Let η be a primitive root of degree p^3 of 1; let η be the matrix corresponding to the operator of multiplication by η in the I_p-basis $1, \eta, \dots, \eta^{\varphi(p^3)-1}$ of the ring $I_p[\eta]$, and let m be an arbitrary natural number.

Then, for $p > 2$, the representations of the following series will be indecomposable:

$$a \to \begin{pmatrix} (\tilde{\eta} \times E_m) & ((t) \times E_m) & ((1) \times D_m) \\ 0 & (\tilde{\xi} \times E_m) & ((t) \times E_m) \\ 0 & 0 & (\tilde{\varepsilon} \times E_m) \end{pmatrix}, \tag{3}$$

where $\times$ is the sign of Kronecker multiplication; D_m is a Jordan box of size m with ones on the main diagonal; (t) denotes a matrix in which only the last column is nonzero, and this column, depending on the position of the matrix (t) in the matrix (3), corresponds either to the element $(1-\eta)^{t-1} \in I_p[\eta]$, or to the element $(1-\xi)^{t-1} \in I_p[\xi]$; E_m is the identity matrix.

In the representations of series (3) only three irreducible components $\tilde{\eta}$, $\tilde{\xi}$, and $\tilde{\varepsilon}$ of the group H participate. If $p = 2$, then in order to construct an infinite series of indecomposable representations of the group H, it is necessary to involve all irreducible components of this group over the ring I_2.

Theorem 3. *A finite group G has a finite number of indecomposable representations over the ring of integral p-adic numbers I_p if and only if a Sylow p-subgroup of this group is a cyclic group whose order is not divisible by p^3.*

Theorem 4. *A finite group G whose order is divisible by the cube of a prime number possesses indecomposable representations of arbitrarily large degree over the ring of rational integers Z.*

Theorem 5. *The number of indecomposable representations of a finite cyclic group over the ring Z is finite if and only if its order is free of cubes.*

Theorem 6. *A finite group with square-free order (not necessarily commutative) has only a finite number of indecomposable representations over the ring Z (Z is the ring of rational integers).*

In Reiner' s paper ([5]) the question was posed: is the decomposition of a representation of a finite group over the ring R_p of p-adic rational numbers into a sum of indecomposable representations over this ring always unique?

A negative answer to this question already arises for p-adic representations of cyclic groups of square-free order. The indecomposable I_p-components of a representation Γ over R_p determine this representation up to R_p-equivalence, but these components may be combined in different ways into R_p-components of the representation Γ. The integral forms of the representations of series 10) for the cyclic group of order p^2 ($p > 2$) make it possible to give a negative solution to Reiner' s second problem ([6]): can nonequivalent Z-representations of a finite group G become equivalent after extending the ring Z to the ring of all integral elements of some field of algebraic numbers? Let $p = 4m + 1$,

* In particular, (1) denotes a matrix of the form $\binom{1}{0}$.

and α is a quadratic nonresidue modulo p. We form the Z-representation $\Gamma_t(\alpha)$ of the cyclic group (a) $(a^{p^2} = 1)$ by replacing the column

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

by the column

$$\begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(see series 10). Then the representations $\Gamma_t(1)$ and $\Gamma_t(\alpha)$ are not equivalent over the ring Z, but are already equivalent over the ring $Z(\varepsilon)$, where $\varepsilon^p = 1$.

Moreover, there is a Z-equivalence

$$\Gamma(1) + \Gamma(\alpha) = \Gamma(1) + \Gamma(1),$$

and hence, for representations of finite groups over the ring of rational integers, the cancellation condition is not satisfied.

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Note: Figure translations are in progress. See original paper for figures.