HYDROMECHANICS

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Abstract

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ON THE ASYMPTOTICS OF WAVES IN A LIQUID OF FINITE DEPTH, CAUSED BY AN ARBITRARY INITIAL ELEVATION OF THE FREE SURFACE

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If the free surface of a heavy liquid had the initial elevation $\eta(x,0)=f(x)$, then, according to the linear theory,

$$\eta(x,t)=\frac{1}{4\pi}\left\{\int_{-\infty}^{\infty}F(s)e^{i(sx-\omega t)}\,ds+\int_{-\infty}^{\infty}F(s)e^{i(sx-\omega t)}\,ds\right\}; \qquad (1)$$

$$\omega=\sqrt{s\ \mathrm{th}\, s}\ \mathrm{sgn}\, s;$$

$F(s)$ is the Fourier transform of the function $f(x)$; the depth h is taken as the unit of length, and $\sqrt{h/g}$ as the unit of time (g is the acceleration of gravity).

We shall consider waves propagating to the right from the place of disturbance ($x>0$); then the second integral in (1) will be small in comparison with the first. We shall denote the first integral by I; in it we replace $\omega(s)$ and $F(s)$ by their approximate values in a neighborhood of zero,

$$\omega_0=s-\frac{s^3}{6}, \qquad F_0(s)=|s|^p(a+ib\ \mathrm{sgn}\, s),$$

and denote the resulting expression by I_0.

The asymptotic formulas for $\eta(x,t)$, obtained by the method of stationary phase as $t\to\infty$, are unsuitable for describing waves moving with maximal velocity, but the idea of the method suggests taking I_0 as the asymptotic expression in this case. In work (^{1}), for example, I_0 (in the case $p=1/2$) was used for calculating the profile of the very first wave in a small neighborhood of its crest. But it turns out that the range of applicability of I_0 is much wider; namely, the following assertion is valid:

If $F(s)$ in $(-\infty,\infty)$ has absolutely integrable derivatives up to order m, inclusive, and in $(-\infty,-\Delta)$ and (Δ,∞), for arbitrarily small Δ, up to order n, where $n\geq p/3+2$, and for $0<|s|\ll 1$,

$$\left|F^{(k)} - F_0^{(k)}\right| \leq \text{const.}\, |s|^{q-k}, \qquad q \geq p+2 \geq 2, \qquad k = 0, 1, \dots, n, \tag{2}$$

then for values of x

$$x - t \geq -\frac{c^2}{2}\, t^{1/3} \tag{3}$$

the inequality holds

$$t^{\frac{p+1}{3}} \left|\eta(x,t) - \frac{1}{2}\left(\frac{2}{t}\right)^{-\frac{p+1}{3}} \{aA_p(\xi) + bB_p(\xi)\}\right| \leq \frac{C(F)}{1-c^2}\left(t^{-\varepsilon} + t^{-\varepsilon_0}\right), \tag{4}$$

$$\varepsilon_0 = m - \frac{p+1}{3}, \qquad \xi = (x-t)\left(\frac{2}{t}\right)^{1/3},$$

where the integrals

$$A_p(\xi) = \frac{1}{\pi}\int_0^\infty \sigma^p \cos\left(\xi\sigma + \frac{\sigma^3}{3}\right) d\sigma, \qquad B_p(\xi) = \frac{1}{\pi}\int_0^\infty \sigma^p \sin\left(\xi\sigma + \frac{\sigma^3}{3}\right) d\sigma$$

are to be understood in the Abel sense if they diverge; ε and ν are related by

$$\varepsilon = \frac{2}{3} - \frac{p+6}{2}\,\nu.$$

Let us explain that in I_0 we made the change of integration variable

$$s = \left(\frac{2}{t}\right)^{1/3} \sigma.$$

We introduce the notation

$$\varphi = \frac{x}{t}s - \omega, \quad \varphi_0 = \frac{x}{t}s - \omega_0, \quad DF = \frac{d}{ds}\frac{F}{\varphi'}, \quad D_0F = \frac{d}{ds}\frac{F}{\varphi_0'},$$

and outside $[-\delta, \delta]$ integrate I by parts n times:

$$4\pi I = \int_{|s|\leqslant\delta} Fe^{i\varphi t}\, ds + \sum_{k=0}^{n-1} \frac{1}{\varphi'}\left(\frac{i}{t}\right)^{k+1} e^{i\varphi t} D^{kF}\Big|_{-\delta}^{\delta} + \left(\frac{i}{t}\right)^n \int_{\delta\leqslant|s|} e^{i\varphi t} D^{nF}\, ds; \tag{5}$$

replacing in this expression F, ω by F_0, ω_0, denoting the result by I_0^*, we estimate the difference

$$
\begin{aligned}
4\pi|I - I_0^*| \leqslant & \left|\int_{|s|\leqslant\delta} (Fe^{i\varphi t} - F_0 e^{i\varphi_0 t})\, ds\right| + \sum_{k=0}^{n-1} \left|\frac{e^{i\varphi t} D^{kF}}{\varphi' t^{k+1}} - \frac{e^{i\varphi_0 t} D_0^k F_0}{\varphi_0' t^{k+1}}\right|\Bigg|_{\pm\delta} \\
& + \frac{1}{t^n}\left|\int_{\delta\leqslant|s|\leqslant\delta_1} (e^{i\varphi t} D^{nF} - e^{i\varphi_0 t} D_0^n F_0)\, ds\right| \\
& + \frac{1}{t^n}\left|\int_{\delta_1\leqslant|s|} e^{i\varphi t} D^{nF}\, ds\right| + \frac{1}{t^n}\left|\int_{\delta_1\leqslant|s|} e^{i\varphi_0 t} D_0^n F_0\, ds\right|.
\end{aligned}
\tag{6}
$$

On the numbers $\delta_0, \delta, \delta_1$ we impose the restrictions

$$
0 < \delta \leqslant \delta_1 \leqslant 1, \qquad \frac{\delta_0}{\delta} \leqslant c < 1, \qquad \delta^3 t \geqslant 1 \tag{7}
$$

and we shall assume

$$
x \geqslant \omega'(\delta_0)t;
$$

then in the interval $\delta \leqslant |s| \leqslant 1$, noting that $0 < \omega'(s) \leqslant 1$ is even and is maximal at $s = 0$, we have

$$
\varphi' \geqslant (1 - \omega'(1))(1 - c^2)s^2, \tag{8}
$$

and for $|s| \geqslant 1$,

$$
\varphi' \geqslant (1 - \omega'(1))(1 - c^2). \tag{8a}
$$

Taking this into account, we estimate $D^{rF},\ D^{rF} - D_0^r F_0$. In detail,

$$
D^{rF} = \sum_{\substack{k_0+\ldots+jk_j=r \\ k_1+\ldots+k_j=k-r}} \frac{F^{(k_0)}(\varphi'')^{k_1}\ldots(\varphi^{(j+1)})^{k_j}}{(\varphi')^k};
$$

as in the other intermediate calculations, the constant coefficients are omitted here. By virtue of (8a) and the boundedness of $\omega^{(k)}(s)$, we have, for $|s| \geqslant 1$,

$$
|D^{rF}| \leqslant \frac{1}{1 - c^2} \sum_{k=0}^{r} |F^k|, \tag{9}
$$

and for $\delta \le |s| \le 1$, taking into account that $\varphi^{(2k)} = -\omega^{(2k)} \sim s$ $(k = 1, 2, \ldots)$, and using (8), we obtain

$$|D^{rF}| \le \frac{1}{1-c^2} \sum |s|^{p-k_0-2k+k_1+k_3+\cdots} \le \frac{1}{1-c^2} |s|^{p-3r}, \tag{10}$$

since, expressing k and k_0 in terms of the remaining summation indices, we have

$$p - k_0 - 2k + k_1 + k_3 + \cdots = p - 3r + 2k_3 + \cdots \ge p - 3r. \tag{11}$$

It is not difficult to see that $D_0^r F_0$ satisfies inequality (10) for all $|s| \ge \delta$, since in this case, because the derivatives of φ_0 above third order are equal to zero $(k_3 = \cdots = 0)$, equality always holds in (11).

Thus, $D_0^r F_0$ is absolutely integrable for $r > (p+1)/3$, and D^{rF} for $r \le n$; therefore the integration by parts carried out in (5) is permissible.

By analogous, but longer, calculations, using (2), (8), and the obvious fact

$$|s| \le 1, \qquad |\varphi^{(k)} - \varphi_0^{(k)}| \le \mathrm{const} \cdot |s|^{5-k}, \qquad k = 0, 1, \ldots, \tag{12}$$

we obtain

$$\delta \le |s| \le 1, \qquad |D^{rF} - D_0^r F_0| \le \frac{1}{1-c^2} \left(|s|^{p+2-3r} + |s|^{q-3r} \right). \tag{13}$$

Now let us return to (6). In view of (2) and (12),

$$|Fe^{i\varphi t} - F_0 e^{i\varphi_0 t}| = |(F - F_0)e^{i\varphi t} + F_0(e^{i\varphi t} - e^{i\varphi_0 t})| \le |s|^q + t|s|^{p+5}.$$

Proceeding similarly with the remaining terms, taking into account the inequalities (10) and (13) obtained above, and simplifying with the help of the last of inequalities (7), we obtain

$$|I - I_0^*| \le \frac{C}{1-c^2} \left\{ \delta^{q+1} + t\delta^{p+6} + \left(t\delta_1^{p+6} + \delta_1^{p+1} \right) (t\delta_1^3)^{-n} + t^{-n} \right\}. \tag{14}$$

Next we integrate by parts the second integral in (1) m times; then

$$\left| \int_{-\infty}^{\infty} F e^{i(sx+\omega t)}\, ds \right| \le C t^{-m}, \tag{15}$$

where the constant C in these inequalities depends only on the function $F(s)$. Now, varying δ, δ_1 subject to the conditions (7), we minimize the right-hand side of (14). Put $\delta_0 = c\delta$, $\delta = t^{-\alpha}$ $(\alpha \le 1/3)$, $\delta_1 = t^{-\alpha_1}$ $(\alpha \ge \alpha_1 \ge 0)$; for brevity we restrict ourselves to the case $n \geq p/3+2,\ q \geq p+2$ (the result is valid for $n > (p+1)/3,\ q > p$); then the first term may be neglected in comparison with the second, while the third is minimal for $\alpha_1 = 0$ and is $\leq t^{-n+1}$, and the second term is $\geq t^{-(p+3)/3} \geq t^{-n+1}$.

Thus, on the right-hand side of (14) only the second term may be retained, since the remaining terms can be made smaller than its minimal value. Combining (14) and (15), and denoting $1-2\alpha = 1/3+\nu,\ \alpha(p+6)-1 = (p+1)/3+\varepsilon$, we complete the proof of the assertion if we show that

$$I_0^* = \lim_{w\to+0}\int_{-\infty}^{\infty} e^{-w|s|}F_0 e^{i\varphi_0 t}\,ds. \tag{16}$$

This is obvious if I_0 exists in the ordinary sense; in the opposite case—

where $I_0(w)$ $(w>0)$ is first integrated by parts analogously to (5):

$$4\pi I_0(w) = \int_{|s|\leqslant 1} e^{-w|s|}F_0 e^{i\varphi_0 t}\,ds + \sum_{k=0}^{n-1}\frac{1}{\varphi_0'}\left(\frac{i}{t}\right)^{k+1} e^{i\varphi_0 t} D^k\left(e^{-w|s|}F_0\right)\Bigg|_{-1}^{1} + \left(\frac{i}{t}\right)^{n}\int_{1\leqslant|s|} e^{i\varphi_0 t} D_0^n\left(e^{-w|s|}F_0\right)ds. \tag{17}$$

Let us verify that the last integral converges uniformly with respect to $w \geqslant 0$; indeed,

$$\left|e^{i\varphi_0 t}D^n\left(e^{-w|s|}F_0\right)\right| \leqslant |s|^{p-3n}\sum_{k=0}^{n}|ws|^k e^{-w|s|} \leqslant \left(1+\sum_{k=1}^{n}k^k e^{-k}\right)|s|^{p-3n}.$$

Passing to the limit in expression (17), we obtain (16), since in I_0^* one may put $\delta = 1$, because it in fact does not depend on δ.

The functions $A_p(x)$, $B_p(x)$ satisfy the differential equation

$$y''' - xy' - (p+1)y = 0$$

and in certain special cases can be expressed in terms of the well-known Airy functions (2),

$$A_{2k}(x) = (-1)^k \operatorname{Ai}^{(2k)}(x), \qquad B_{2k+1}(x) = (-1)^{k+1}\operatorname{Ai}^{(2k+1)}(x), \qquad k = 0,1,\dots.$$

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Note: Figure translations are in progress. See original paper for figures.