PHYSICS

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1962

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Abstract

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QUASILINEAR EQUATIONS FOR A QUANTUM PLASMA

(Presented by Academician M. A. Leontovich, 21 VI 1962)

1. Recently, a number of works have considered the quasilinear theory of plasma, used to describe processes occurring in a weakly turbulent plasma. In the present article a derivation is given of quasilinear equations for a quantum plasma, which may prove useful in studying the weakly turbulent state of a carrier plasma in a solid (see ([1], [2])).

As in the case of a classical plasma, we shall start from equations with a self-consistent field φ; here, for the density matrix in the Wigner representation,

$$f_{xp} = \sum_{\xi} e^{-i\xi p} \rho \left(x - \frac{\xi}{2}, x + \frac{\xi}{2} \right),$$

where $\rho(y, z)$ satisfies the equation*

$$i \frac{\partial \rho(y,z)}{\partial t} = \left[-\frac{\Delta_y}{2} + \frac{\Delta_z}{2} + e\varphi(y) - e\varphi(z) \right] \rho(y,z) =$$

$$= \left[\nabla_x \nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right) \right] \rho\left(x + \frac{\xi}{2}, x - \frac{\xi}{2}\right),$$

we have:

$$\frac{\partial f_{xp}}{\partial t} = \frac{1}{i} \sum_{\xi} e^{-i\xi p} \left[\nabla_x \nabla_\xi + e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right) \right] \sum_q e^{i\xi q} f_{xq} =$$

$$= -p \frac{\partial f_{xp}}{\partial x} + \frac{1}{i} \sum_{\xi q} \left[e\varphi\left(x + \frac{\xi}{2}\right) - e\varphi\left(x - \frac{\xi}{2}\right) \right] e^{i\xi(q-p)} f_{xq}. \quad (1)$$

Equation (1), together with Poisson' s equation**

$$\Delta_x \varphi = 4\pi n e \left(\sum_p f_{xp} - 1 \right) \quad (2)$$

in the quasilinear theory of plasma is replaced by a system of equations for the averaged*** value of the quantum distribution function $f^0 = \langle f_{xp} \rangle$ and for the os-

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* We put $\hbar = m = 1$ and, for simplicity, consider below a quadratic isotropic spectrum $\varepsilon_p = p^2/2$.

** For simplicity, the case of longitudinal oscillations of an electron plasma with a positive "background" of space charge is considered.

*** Averaging is carried out over a time interval significantly exceeding the period of plasma oscillations.

the time-oscillating deviation of the distribution function f_{xp} from its mean value (this deviation is assumed to be small).*

2. Separating in (1)–(2) the oscillating terms and passing to spatial Fourier components,

$$\varphi(x) = \sum_k \varphi_k e^{ikx}, \qquad f_{xp} - \langle f_{xp} \rangle = \sum_k f^1_{kp} e^{ikx}, \tag{3}$$

we obtain, for the spatially homogeneous case ($\nabla_x f^0 = 0$),

$$\dot{f}^1_{kp} + ikp f^1_{kp} = e\varphi_k \frac{f^0_{p+k/2} - f^0_{p-k/2}}{i}, \qquad \varphi_k = -4\pi n e k^{-2} \sum_p f^1_{kp}. \tag{4}$$

On the other hand, carrying out the averaging over x in (1), we have:

$$\frac{\partial f^0_p}{\partial t} = i \sum_k e\varphi^+_k \left[f^1_{k,\,p+k/2} - f^1_{k,\,p-k/2} \right]. \tag{5}$$

Next, integrating the ordinary differential equation (4) for the oscillating part of the density matrix f^1_{kp}, we obtain:

$$f^1_{kp}(t) = f^1_{kp}(0) e^{-ikpt} + \int_0^t e\varphi_k(t') \, \frac{f^0_{p+k/2}(t') - f^0_{p-k/2}(t')}{i} \, e^{-ikp(t-t')} \, dt'. \tag{6}$$

We now differentiate (4) twice with respect to time, multiply by $\dot{\varphi}^+_k$, and, adding the complex-conjugate expression, substitute f^1_{kp} from (6):

$$d\left(|\dot{\varphi}_k|^2 + \omega_k^2 |\varphi_k|^2\right)/dt =$$

$$= 4\pi ne^2 k^{-2}\dot{\varphi}_k^+(t)\sum_p (kp)^2\left\{f_{kp}^1(0)e^{-ikpt}+\right.$$

$$\left.+\int_0^t e\varphi_k(t')\,\frac{f_{p+k/2}^0 - f_{p-k/2}^0}{i}\,e^{-ikp(t-t')}\,dt'\right\} + \text{c. c.},$$

$$\omega_k^2 = 4\pi ne^2\sum_p \frac{kp}{k^2}\left(f_{p+k/2}^0 - f_{p-k/2}^0\right) \simeq 4\pi ne^2. \tag{7}$$

Substituting into (7) $\varphi_k(t) = |\varphi_k(t)|e^{-i\omega_k t}$, we find

$$2\frac{d}{dt}\left(|\varphi_k|^2\omega_k^2\right) = -4\pi ne^2\omega_k\sum_p \frac{(kp)^2}{k^2}|\varphi_k(t)|\times$$

$$\times\int_0^t |\varphi_k(t')|\left(f_{p+k/2}^0(t') - f_{p-k/2}^0(t')\right)e^{-ikp(t-t')}\,dt' + \text{c. c.}$$

We regard the slowly time-varying functions $|\varphi_k(t)|$ and $f^0(t)$ as constant; taking them outside the integral sign with respect to t' and taking into account that

$$\int_0^t e^{i\alpha(t-t')}\,dt' + \text{c. c.} \underset{t\to\infty}{\longrightarrow} 2\pi\delta(\alpha),$$

we obtain

$$\frac{d}{dt}\,\omega_k^2|\varphi_k|^2 = 4\pi ne^2|\varphi_k|^2\omega_k\sum_p \frac{(kp)^2}{k^2}\left(f_{p+k/2}^0 - f_{p-k/2}^0\right)\pi\delta(\omega_k - kp). \tag{8}$$

* The spectrum of linear oscillations of a quantum plasma has been investigated in a number of works (see, for example, (3), where the literature is given).

Similarly, from equation (5), after substituting f_{kp}^1 from (6) into this equation, we obtain

$$\frac{df_p^0}{dt} = +\pi e^2\sum_k |\varphi_k|^2\left\{\left(f_{p+k}^0 - f_p^0\right)\delta\left(\omega_k - k\left(p+\frac{k}{2}\right)\right) - \right.$$

$$\left. - \left(f_p^0 - f_{p-k}^0\right)\delta\left(\omega_k - k\left(p - \frac{k}{2}\right)\right)\right\}. \tag{9}$$

3. Equations (8)–(9) form a closed system of quasilinear equations for a quantum plasma. This system, obtained (under certain assumptions) from the equations with a self-consistent field, contains less information than the original equations (from (8) and (9) we can, for example, find only the amplitudes, but not the phases, of the fields; the field amplitudes themselves are assumed to be sufficiently small–otherwise the equations are invalid–and so on), but this deficiency is compensated by their simplicity. Indeed, equations (8)–(9) have the form of kinetic equations for an almost ideal system of "waves" –bosons and "particles" –fermions, if the only process occurring in such a system is emission (absorption) of a "wave" by a "particle," and if the density of "waves" in phase space N_k is large, so that the matrix elements of the absorption and emission processes are proportional to $\sqrt{N_k}$. Then, for the change in the number of "particles" F_p in momentum space p, caused by these processes, we have:

a) "loss" due to absorption:

$$-\sum_k F_p N_k W_{p,p+k}\, \delta(\varepsilon_p + \omega_k - \varepsilon_{p+k});$$

b) "loss" due to radiation:

$$-\sum_k F_p N_k W_{p,p-k}\, \delta(\varepsilon_p - \omega_k - \varepsilon_{p-k});$$

c) "gain" due to absorption:

$$\sum_k F_{p-k} N_k W_{p-k,p}\, \delta(\varepsilon_{p-k} + \omega_k - \varepsilon_p);$$

d) "gain" due to radiation:

$$\sum_k F_{p+k} N_k W_{p+k,p}\, \delta(\varepsilon_{p+k} - \omega_k - \varepsilon_p)$$

($W_{p',p} = W_{p,p'}$ is the probability of absorption with transition of a "particle" from state p to state p'; $\varepsilon_p = p^2/2$ is the energy of the "particle"; ω_k is the energy of the "wave"). Summing the contributions a)–d), we obtain the equation for F_p

$$\frac{\partial F_p}{\partial t} = \sum_k W_{p,p+k} N_k \left\{ (F_{p+k} - F_p)\delta\left(\omega_k - k\left(p + \frac{k}{2}\right)\right) - \right.$$
$$\left. -(F_p - F_{p-k})\delta\left(\omega_k - k\left(p - \frac{k}{2}\right)\right)\right\}. \tag{10}$$

Similarly, for the change in the density of "waves" N_k, caused by the same processes, we have

$$\frac{\partial N_k}{\partial t} = N_k \sum_p W_{p+k/2,p-k/2} \left(F_{p+k/2} - F_{p-k/2}\right) \delta(\omega_k - kp). \tag{11}$$

From comparison of (10)–(11) with (8)–(9) it is seen that these systems coincide if we set:

$$W_{p,p'} = 4\pi^2 e^2 \frac{\omega_{p-p'}}{|p-p'|^2}, \qquad N_k = \frac{k^2|\varphi_k^2|}{4\pi\omega_k}.$$

Thus, the quasilinear equations (8)–(9) indeed have the form of a system of kinetic equations for "particles" and "waves" in the case when the density of the "waves" (which are collective oscillations of the plasma) is so large that the contribution of the processes of spontaneous emission of "waves" by "particles" (as well as of pair collisions of "particles" with one another) is small in comparison with the contribution of the processes of induced emission and absorption.

Let us finally point out the following circumstance: the structure of the equations obtained is such that the conservation laws for the energy

$$\sum_p \varepsilon_p F_p + \sum_k \omega_k N_k$$

and momentum

$$\sum_p pF_p + \sum_k kN_k$$

of the system of "waves" and "particles" are automatically satisfied.

Received
10 IV 1962

References

[1] A. R. Hutson et al., Phys. Rev. Lett., **7**, 237 (1961). [2] L. Esaki, Phys. Rev. Lett., **8**, 4 (1962). [3] V. P. Silin, A. A. Rukhadze, *Electromagnetic Properties of Plasma and Plasma-like Media*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.