MATHEMATICS

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Abstract

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ON OPEN AND ALMOST-OPEN MAPPINGS OF TOPOLOGICAL SPACES

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§ 1. I first encountered the concept of an almost-open mapping in P. Vopenka. There, certain results on almost-open mappings were also given, in particular the theorem:

An arbitrary bicompactum is an almost-open image of some zero-dimensional bicompactum of the same weight.

Below an attempt is made to set forth systematically the principal properties of almost-open mappings. A number of theorems is proved; some of them are simply transfers of results proved earlier for open mappings. On the other hand, a large group of the theorems presented ceases to be true if one speaks of open mappings. We consider only continuous mappings.

Basic definition 1. A mapping f of a space X onto Y is called **inductively open** if there exists an $X_1 \subseteq X$—a subspace of the space X—such that $fX_1 = Y$ and the mapping $f : X_1 \to Y$ is open on X_1.

Properties of inductively open mappings:

1. If $f : X \to Y$ is inductively open and $Y_1 \subseteq Y$ is an arbitrary subspace of the space Y, and $X_1 = f^{-1}Y_1$, then on X_1 the mapping $f : X_1 \to Y_1$ is inductively open.
2. The superposition of a finite number of inductively open mappings is an inductively open mapping; if $f_1 : X \to Y$ is an inductively open mapping and $f_2 : Y \to Z$ is an inductively open mapping, then $f_2 f_1 : X \to Z$ is an inductively open mapping.
3. If $f : X \to Y$ is a continuous mapping, and on some $X_1 \subseteq X$, $fX_1 = Y$, the mapping f is inductively open, then the mapping f is inductively open also on X.
4. If

$$X = \prod_{\alpha \in M} X_\alpha, \quad Y = \prod_{\alpha \in M} Y_\alpha$$

and all $f_\alpha : X_\alpha \to Y_\alpha$ are inductively open, then the mapping

$$f = \prod_{\alpha \in M} f_\alpha, \quad f : X \to Y$$

is also inductively open, where

$$f(x) = f\{x_\alpha\} = \{f_\alpha x_\alpha\} = \{y_\alpha\} = y.$$

5. Let $\gamma = \{X_\alpha\}$ be a family of subspaces of the space X such that $fX_\alpha = Y$ and f is open on X_α. Then their union

   $$\widetilde{X} = \bigcup X_\alpha$$

   also has this property. In particular, if $\gamma = \{X_\alpha\}$ is the family of all subspaces of the space X which f maps openly onto Y, then $\widetilde{X}$ is the maximal subspace of the space X possessing this property (i.e. on no $X' \supset \widetilde{X}$, $X' \neq \widetilde{X}$, will the mapping f be open).

Definition 2. A mapping f of a space X onto Y will be called **almost open** if for an arbitrary point $y \in Y$ there is a point $x \in f^{-1}y$ having a base of open sets, each element of which f maps onto a set open in Y. In this case we shall call the point x a point of **almost-openness** of the mapping f.

There exist examples of almost-open but not inductively open mappings, and of inductively open but not open mappings.

The set of points of the space X satisfying condition 2 of the definition will be called the **set of points of almost-openness** of the mapping f and will be denoted by $\widetilde{X}'$. Clearly, $\widetilde{X} \subseteq \widetilde{X}'$.

It is easy to give an example of an almost-open mapping onto an arbitrary space Y of some (depending on Y) space X such that the set of points of almost-openness of the mapping f is discrete and on it the mapping f is a condensation.

6. If $f : X \to Y$ is an arbitrary continuous mapping and $y \in Y$ is any point of the space Y, then the sets $f^{-1}y \cap \widetilde{X}$, $f^{-1}y \cap \widetilde{X}'$ are closed in $f^{-1}y$.
7. If f is an arbitrary almost-open mapping, $fX = Y$, and ξ is an open cover of the space X, then the system $\eta = \langle \xi \rangle$, consisting of the interiors of all sets of the form fG, where $G \in \xi$, forms a cover of the space Y. Conversely, if this condition is satisfied for an arbitrary open cover ξ of the space X, then f is almost-open.

Thus, the following inclusions hold for classes of mappings: open $\subseteq$ inductively open $\subseteq$ almost-open*.

§ 2. **Results.** All the theorems of this section cease to be true if the word "inductively open" is replaced by "open."

Main theorem 1. *An arbitrary metric space is an inductively open, closed, bicompact (and continuous) image of a zero-dimensional, in the sense of* dim, *metric space of the same weight.*

From the main theorem the following theorems follow:

Theorem 2. *An arbitrary compactum is an inductively open (closed, bicompact and continuous) image of some zero-dimensional compactum.*

Theorem 3. *The segment* $I^1 = [0, 1]$ *is an inductively open image of the Cantor perfect set.*

From theorem 3, the properties of inductively open mappings collected in § 1, and the well-known theorem of Tikhonov (1), many consequences follow:

Theorem 4. *The cube* I^τ *(of weight* τ*) is an inductively open image of* D^τ*.*

Theorem 5. *An arbitrary completely regular space* Y *is an inductively open, closed, bicompact image of some zero-dimensional in the sense of* ind, *completely regular space* X *of the same weight.*

Theorem 6. *An arbitrary countably paracompact space* Y *is an inductively open, closed, bicompact image of some zero-dimensional in the sense of* ind, *countably paracompact space* X *of the same weight.*

Theorem 7. *An arbitrary paracompact space Y is an inductively open, closed, bicompact image of some zero-dimensional** paracompact space X.*

Theorem 8. *An arbitrary bicompactum of weight* τ *is an inductively open image of a zero-dimensional bicompactum of the same weight***.*

In connection with theorems 5-8 it is necessary to point out results obtained earlier by V. Ponomarev (5): he proved assertions which are obtained from the formulations of theorems 5-8 by replacing the word "inductively open" by the word "irreducible." It is evident, however, that the properties of inductive openness and irreducibility are incompatible, since an open mapping is always reducible.

§ 3. **Other results.** The results of this section show that very many properties of open mappings are inherent in inductively open and even almost-open mappings.

* Properties 1-4 are also possessed by almost-open mappings.

** I.e. $\dim X = 0$. If one requires that the weight of X be no greater than the weight of Y, then I can prove only that $\operatorname{ind} X = 0$.

*** For almost-open mappings this theorem was proved earlier by P. Vopenka.

Theorem 9. Under almost-open mappings:

1. The weight of a space does not increase.

2. The character of a space does not increase.

In particular, a space with the first axiom of countability is mapped into a space with the first axiom of countability.

3. A point-countable base is preserved.

Theorem 10. Under an almost-open compact mapping, a metric space is mapped into a space with a uniform base.

Theorem 11. Under an almost-open closed mapping of a metric space onto a T_1-space, the image is metrizable.

Theorem 12. Under an almost-open regular mapping, the image of a metrizable space is metrizable (provided it has the T_1-axiom of separability) (see ([1])).

Theorem 13. Under an almost-open mapping, a complete topological space is mapped into a complete one.

Lemma. If f is an inductively open mapping of a metric space, then the set $\widetilde{X}$ of points of openness of this mapping is a set of type G_δ in X.

With the aid of this lemma one proves

Theorem 14. Every complete metric space is an open, compact image of some zero-dimensional (in the sense of dim) complete metric space of the same weight.

Michael’ s remarkable theorem can be formulated as follows:

Theorem 15. Let $f : X \to Y$ be an inductively open mapping of a metric space X onto a paracompact space Y, under which all inverse images of points of Y are complete metric spaces.

Then there is an $X_1 \subseteq X$ such that, on X_1, the mapping f is: 1) inductively open; 2) closed; 3) bicompact; 4) $fX_1 = Y$. The space Y therefore turns out to be metrizable.

Corollary. A paracompact space which is an inductively open image of a complete metric space is metrizable.

§ 4. **Relation to dimension.**

Theorem 16. An almost-open finite-to-one image of a weakly countable-dimensional space is a weakly countable-dimensional space.

Theorem 17. A metric space which is an almost-open finite-to-one image of a countable-dimensional metric space is countable-dimensional (see ([3])).

We see that, in the case of finite-to-one mappings, almost-open mappings behave with respect to dimension almost like open mappings. If the finite multiplicity of the mapping is abandoned, then there is no analogy with open mappings in this sense, as we saw in § 2.

Theorem 18. An inductively open, countable-to-one mapping does not raise the dimension of complete metric spaces with a countable base (finite-dimensional, weakly countable-dimensional, or countable-dimensional) (see (^{3})).

Theorem 19. An open, closed countable-to-one mapping does not raise the dimension of metric spaces (finite-dimensional or weakly countable-dimensional).

In connection with these results one should point to the well-known theorem of P. S. Aleksandrov (^{2}), which can now be generalized as follows:

Corollary. An inductively open countable-to-one mapping does not raise the dimension of a compactum.

At the basis of the proofs of Theorems 18 and 19 lie the aforementioned properties of inductively open mappings, the results of A. D. Taimanov, who proved Theorem 18 under the assumption that the mapping is open and Theorem 19 under the assumption that there is a countable base in X (^{6}), and also the lemma:

A point-countable conservative sum of closed k-dimensional spaces is k-dimensional.

Nagami (^{8}) proved that under an open finite-multiple mapping of a finite-dimensional paracompact space onto a paracompact space, dimension does not change. A direct generalization of this result is the following

Theorem 20. *Under an open finite-multiple mapping of a normal finite-dimensional or weakly countably-dimensional space onto a weakly paracompact normal space, dimension does not change.*

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Note: Figure translations are in progress. See original paper for figures.