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Abstract

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GRADIENT INVARIANCE AND THE AXIOMATIC APPROACH IN QUANTUM FIELD THEORY

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1. Let $\varphi_1, \ldots, \varphi_n$ be Hermitian operators describing quantum fields (both with integral and with half-integral spin), for which the quantum-mechanical axioms (I), the axiom of relativistic invariance (II), the axioms of positivity of energy (III) and uniqueness of the vacuum (IV) are satisfied. The electromagnetic field A_μ is not included among the φ_j. Let us add the condition that the theory possesses gradient invariance, i.e. invariance under the substitution

$$A_\mu \to A_\mu + \partial f/\partial x_\mu, \qquad \varphi_j \to \varphi_j + ieq_j f \varphi_i, \tag{1}$$

where eq_j is the electric charge; q_j is a Hermitian antisymmetric matrix; the infinitesimally small operator f is an arbitrary function of x.

Let us consider the question: is it possible, starting from axioms (I)–(IV), to obtain as a result of the transformation (1) an electric-charge operator different from zero? In other words: does the possibility of the existence of charged particles and the rule for constructing the charge operator follow solely from axioms (I)–(IV) and gradient invariance?

2. Let us turn to the Wightman functions W_φ ([1]), or the vacuum expectation values of products of field operators $W_\varphi\{x\} = \langle 0|\varphi_1(x_1)\ldots\varphi_n(x_n)|0\rangle$, the complete set of which determines the state of the quantum field. The functions $W_\varphi\{x\}$ are translationally invariant; let us choose the differences in the form $\xi_j = x_{j+1} - x_j$; according to axioms (I), (III), and (IV), $W_\varphi\{x\}$ is the boundary value of a function $W_\varphi\{z\}$, $z = x - iy$, analytic in the domain T_n, consisting of points $\zeta_1 = z_2 - z_1, \ldots, \zeta_{n-1}$, $\zeta = \xi - i\eta$, $\eta_0^2 - \vec{\eta}^{\,2} > 0$, $\eta_0 > 0$. Taking axiom (II) into account, the domain of analyticity can be extended to the domain T'_n, consisting of points $\Lambda_c\zeta_1 \ldots \Lambda_c\zeta_{n-1}$, where Λ_c is a complex proper Lorentz transformation which is the analytic continuation of a real proper Lorentz transformation, $\det \Lambda_c = +1$; the functions $W_\varphi\{z\}$ are invariant with respect to the transformations Λ_c ([2]).

3. Let us introduce the electromagnetic field A_μ. This field can be relativistically invariantly decomposed into two parts $A_\mu^1 + A_\mu^0$, where A_μ^1 describes a field with spin 1, and A_μ^0 a field with spin 0. The field A_μ^1 is not changed under the transformation (1); the field A^0_μ, which is not a dynamical variable and does not obey equations of motion, takes on the entire gradient addition $\partial f/\partial x_\mu$.

The operators A^0_μ act in a space $\mathfrak{H}$ different from the Hilbert space of physical states. The arbitrariness of A^0_μ and their nonphysical character are described, as is known, by introducing states $|m\rangle$ with negative energy or an indefinite metric in the space $\mathfrak{H}$ with matrix elements $\langle m'|A^0_\mu|m\rangle \neq 0$. In view of this, the vacuum expectations $W_\varphi\{x, A^0\}$, containing the fields A^0_μ, cannot be continued into the complex domain T_n with respect to the coordinates ξ_A entering into $\langle m'|A^0_\mu(\xi_A)|m\rangle$.

After continuation into the complex domain with respect to the other variables, the function $W_\varphi\{z, z^*, A^0\}$ will depend on z^* by means of $\xi_A = \zeta_A + \zeta^*_A$; $\zeta_j = z_{j+1} - z_j$.

The question posed by us can now be formulated as follows: the function $W_\varphi\{x, A\}$ is the sum of two parts $W_\varphi\{x, A^0\} + W_\varphi\{x, A^1\}$, where $W_\varphi\{x, A^1\}$ is the boundary value of the function $W_\varphi\{z, A^1\}$, analytic in the domain T'_n, while the function $W_\varphi\{z, z^*, A^0\}$ depends not only on z, but also on z^* by means of $\xi_A = \zeta_A + \zeta^*_A$. Is this circumstance essential in order that one may assert invariance of the theory with respect to phase transformations leading to the appearance of the charge q_j?

Indeed, if f does not depend on the coordinates, then f may be regarded as a parameter, and the invariance of the theory with respect to the transformations $1 + ieQf$ leads to conservation of charge:

$$W_{\varphi+\delta\varphi}\{x, A\} = W_\varphi\{x, A\}; \qquad \sum q_j W_\varphi\{x, A\} = 0. \tag{2}$$

4. The analyticity of $W_\varphi\{z\}$ as a function of the coordinates, by virtue of the translational invariance of the theory, may be expressed by the formula

$$-i\frac{\partial}{\partial z^*_{j\mu}} W_\varphi\{z\} \equiv P^-_\mu(j) W_\varphi\{z\} = 0, \qquad j = 1, 2, \dots, n. \tag{3}$$

The functions $W_\varphi\{z, z^*, A^0\}$ depend both on z and on z^*; therefore one must also consider transformations that do not preserve the domains of analyticity. The generators of such transformations do not commute with P^-_μ; the generators of the analytic part of the complex Lorentz group (see item 2), however, commute with P^-_μ, but not with $P^+_\mu = -i\,\partial/\partial z_\mu$. Thus, starting from the existence of A^0, we have arrived at the conclusion that it is necessary to introduce the complex Lorentz group $L_+(C)$.

Let us denote by $M_{\mu\nu}$ that generator of the inhomogeneous group $L_+(C)$ which, together with P_λ, corresponds to the real subgroup: $[P_\lambda, M_{\mu\nu}] = i[g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu]$. The commutation relations for the inhomogeneous group $L_+(C)$ split into two independent parts, containing the operators $P^{\pm}_{\mu}$ and $M^{\pm}_{\mu\nu} = \frac{1}{2}[M_{\mu\nu} \pm M'_{\mu\nu}]$:

$$[P^{\pm}_{\lambda}, M^{\pm}_{\mu\nu}] = i\,[g_{\lambda\mu}P^{\pm}_{\nu} - g_{\lambda\nu}P^{\pm}_{\mu}],$$

$$[M^{\pm}_{\mu\nu}, M^{\pm}_{\lambda\sigma}] = i\,[g_{\mu\lambda}M^{\pm}_{\nu\sigma} - g_{\nu\sigma}M^{\pm}_{\mu\lambda} - g_{\mu\sigma}M^{\pm}_{\nu\lambda} + g_{\nu\sigma}M^{\pm}_{\mu\lambda}], \tag{4}$$

where either the upper or the lower indices must be taken. The operators $P^{+}_{\lambda}, M^{+}_{\mu\nu}$ commute with the operators $P^{-}_{\sigma}, M^{-}_{\gamma\tau}$. The commutation relations (4) have the same form as in the case of the inhomogeneous real Lorentz group (the inhomogeneous group $L_{+}(C)$, from the point of view of interest to us, is described in (3)).

It follows from relations (4) that, when applied to the functions $W_{\varphi}\{z\}$, analytic with respect to the variables z (condition (3)), the operators $M^{-}_{\mu\nu}$ do not depend on the space-time rotations and displacements, which in this case are determined exclusively by the operators

$$M^{+}_{\mu\nu} \text{ and } P^{+}_{\lambda}: \qquad P^{-}_{\lambda} \simeq 0,\ [P_{\lambda}, M^{-}_{\mu\nu}] \simeq 0,$$

$$[P_{\lambda}, M^{+}_{\mu\nu}] \simeq i g_{\lambda\mu}P_{\nu} - i g_{\lambda\nu}P_{\mu}. \tag{5}$$

The sign $\simeq$ in (5) means that the equality has meaning for functions satisfying condition (3). In this case the generators $M^{-}_{\mu\nu}$ determine an additional linear transformation of the components $W_{\varphi}\{z\}$. The necessity of considering these generators is a consequence of the existence of the nonphysical field A^{0}_{μ}.

5. Let us list the properties of the vacuum functions, using the group $L_{+}(C)$. Let $W_{\varphi}(x)$ refer only to physical fields and not contain A^{0}. By

axiom (II), $W_{\varphi}\{x\}$ is invariant with respect to real proper Lorentz transformations L_{+}; the spectral axioms (III), (IV) make it possible to analytically continue $W_{\varphi}\{x\}$ into the domain T_n, with $W_{\varphi}\{x\}$ being the boundary value of $W_{\varphi}\{z\}$; since the function $W_{\varphi}\{z\}$ is Lorentz-invariant, by virtue of the analyticity of the coefficients of the transformation L_{+} it follows that $W_{\varphi}\{z\}$ is invariant with respect to transformations with $M^{+}_{\mu\nu}$ and P^{+}_{λ}:

$$\sum_{j} M^{+}_{\mu\nu}(j)\, W_{\varphi}\{z\} = 0; \qquad \sum_{j} P^{+}_{\lambda}(j)\, W_{\varphi}\{z\} = 0, \tag{6}$$

thereby the domain of analyticity is enlarged to T'_n, and among the points of T'_n there are also real points (Jost points).

In other words, under space-time transformations the vacuum expectations transform according to M^+-representations of the complex group $L_+(C)$, i.e., physical fields in vacuum expectations behave as M^+-spinors, M^+-vectors, etc.

Let us turn to the possible transformations $W_\varphi\{z, A\}$. Since the coordinates x_A in $W_\varphi\{z, A^0\}$, arising from the field A^0_μ, can only be real, $x_A = z_A + z^*_A$, this function can be invariant only with respect to transformations of the real subgroup $L_+(C)$, with operators $P_\lambda, M_{\mu\nu}$; A^0_σ is a 4-vector in this subgroup (an "M-vector"):

$$D_A = 1 + i\omega^{\mu\nu} M_{\mu\nu}. \tag{7}$$

The application of D_A to W_φ gives, by virtue of (6),

$$D_A W_\varphi\{z\} = \left\{ 1 + i \sum_j M_{\mu\nu}\omega^{\mu\nu} \right\} W_\varphi\{z\} = 0, \tag{8}$$

i.e., invariance with respect to the transformation $1 + iM_{\mu\nu}\omega^{\mu\nu}$, containing only $M_{\mu\nu}$; the generators M_{jk}, $j, k = 1, 2, 3$, produce in this case a phase transformation.

6. Gradient invariance of the theory means, in particular, that $W_\varphi\{x, A^1\}$ must have the same transformation properties as $W_\varphi\{x, A^0\}$ (for real coordinates). Choosing as $\xi_1, \ldots$ the Jost points $\rho_1, \ldots$, we obtain that invariance with respect to a transformation of type (7) is preserved for $W_\varphi\{z, A^1\}$ also in the domain T_n. Thus the field A^1_μ must be simultaneously an M-vector and an M^+-vector.

In the inhomogeneous Lorentz group with generators $M_{\mu\nu}, P_\lambda$, the invariants $W^2 = W_\mu W^\mu$, where $W_\mu = \frac{1}{2}\varepsilon^{\mu\lambda\nu\sigma} M_{\lambda\nu} P_\sigma$, and $W^\lambda n_\lambda$, where the 4-vector n_λ is spacelike, are equal to zero for the field A^0_μ, since the spin of A^0_μ is zero. Assuming the transformation properties of A^0_μ and A^1_μ to be identical (with respect to the real group $P_\lambda, M_{\mu\nu}$), we find that for A^1_μ also $W^2 = 0$ and $W^\lambda n_\lambda = 0$. But the spin properties of the field A^1_μ are determined by the group $M^+_{\mu\nu}, P_\lambda$ with invariants $W^+_\lambda n^\lambda$, $(W^+)^2$, where $W^{+\mu} = \frac{1}{2}\varepsilon^{\mu\lambda\nu\sigma} M^+_{\lambda\nu} P_\sigma$, and here $W^+_\lambda n^\lambda$ is not equal to zero. Hence it follows that $M^-_{\mu\nu}$ cannot vanish for A^1_μ for all μ, ν, i.e. A^1_μ is not a scalar with respect to transformations with $M^-_{\mu\nu}$.

The exact form of the transformation D^0_A for A^1_μ as an M-vector and an M^+-vector may also depend on the number of independent components of A^1_μ; the minimal symmetry corresponds to the fields $A^1_{\mu\, in, out}$ (two independent components). In this case one can choose such a coordinate system where there are only A^1_1 and A^1_2, and then only the generators M_{12}, M_{03} will enter into D^0_A. But the quantities $M^-_{\lambda\nu}$ as applied to $W_\varphi\{z, A^1\}$ are invariant with respect to the M^+-group, i.e., with respect to the Lorentz group for the M^+-vector A^1_μ. Consequently,

$$D^0_A = 1 + i\omega^{\mu\nu} M^+_{\mu\nu} + iM^-_{\overline{12}}\alpha + M^-_{\overline{03}}\beta \tag{9}$$

with real α, β and Hermitian $M^-_{\overline{12}}$, $M^-_{\overline{03}}$; moreover, according to (8), $M^-_{\overline{12}}$ describes a phase transformation. Since, according to (2), the electric charge is associated precisely with such a transformation, the charge operator is

$$Q = M^-_{\overline{12}} + \text{const.} \tag{10}$$

Thus, the charge operator is the operator of rotation about the "3" axis in the space T, which we shall call isospin space. Since under a phase transformation the field A'_μ does not change, the field A^1_μ is the projection of the pseudovector onto the "3" axis in isospin space: $A^1_\mu \sim M^1_{\overline{12}}$.

For the interacting field A^1_μ the number of independent components is increased by one, associated with the Coulomb interaction. Taking $A^1_3 = 0$ in some coordinate system, we obtain invariance with respect to the transformation $1 + iM_{\overline{12}}\alpha + M_{\overline{01}}\beta_1 + M_{\overline{02}}\beta_2$ with real α, β_1, β_2 and Hermitian $M_{\overline{12}}$, $M_{\overline{01}}$, $M_{\overline{02}}$. As before, only $M_{\overline{12}}$ leads to a phase transformation in accordance with condition (9) for A^1_μ as an M^-- and M^+-vector.

7. Suppose that the mass of particles associated with the vector field B_μ is nonzero, but the theory is gradient invariant. The electromagnetic field is not considered.

Put $B_\mu = B^1_\mu + B^0_\mu$, where B^1_μ belongs to a physical field with spin 1, while the operator B^0_μ is not a dynamical variable and corresponds to a field with spin 0. Following the same path as in the case of A_μ, we arrive at the conclusion that the class of possible transformations D_B coincides with D_A; formula (9) also remains unchanged, since the field B^1_μ must simultaneously be an M^+-vector and an M-vector and, owing to $W^2 = 0$ for B^0_μ, cannot be a scalar with respect to $M_{\mu\nu}$-transformations. Suppose that in the case of the field B^1_μ there are three independent components in isospace $B^1_{\mu 1}$, $B^2_{\mu 2}$, $B^3_{\mu 3}$, which (by analogy with A^1_μ) transform respectively as $M^-_{\overline{23}}$, $M^-_{\overline{13}}$, $M^-_{\overline{12}}$. Then D^0_B will contain quantities $M^-_{\overline{jk}}$, $j,k = 1,2,3$, invariant (with respect to transformations of the subgroup M^+):

$$D^0_B = 1 + iM^+_{\mu\nu}\omega^{\mu\nu} + iM^-_{\overline{jk}}\alpha_{jk}; \tag{11}$$

where α_{jk} are real, and $M^-_{\overline{jk}}$ are Hermitian. The additional group $M^-_{\overline{jk}}$ is the group of rotations in three-dimensional isospin space with invariants T^2, T_3. The fields interacting with B_μ can be classified according to the values of isospin

T and the projection $T_3 = M_{\overline{12}}^{-}$. For $T_3 = Q$ only the values $T = 0, 1$ are possible.

The field B_μ possesses isospin (and charge), whereas the electromagnetic field carries no charge. The fields $B_{\mu in,out}$ can be only isovector fields, which follows from $W^2 = 0$ and $(W^+)^2 = 0$ for these fields.

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CITED LITERATURE

1. A. S. Wightman, Phys. Rev., **101**, 860 (1956).

2. S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, N. Y., 1961.

3. Yu. V. Novozhilov, Vestn. LGU, No. 4, 5 (1962).

Note: Figure translations are in progress. See original paper for figures.