V. P. DIDENKO

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Abstract

Full Text

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ON SOME ELLIPTIC SYSTEMS DEGENERATING ON THE BOUNDARY OF A DOMAIN

(Presented by Academician S. L. Sobolev on 18 XII 1961)

In this paper we consider systems of linear partial differential equations of the second order of the form:

$$Lu = -y^m u_{xx} - u_{yy} + a u_x + b u_y + cu = h, \tag{1}$$

$$Lu = -u_{xx} - y^m u_{yy} + a u_x + b u_y + cu = h, \tag{2}$$

where m is a positive real number; $a = \|a_{i,k}\|$, $b = \|b_{i,k}\|$, $c = \|c_{i,k}\|$ are given square matrices of order n; $h = (h_1, \dots, h_n)$ is a given vector; $u = (u_1, \dots, u_n)$ is the unknown vector.

Systems (1) and (2) are strongly elliptic for $y > 0$, with parabolic degeneration for $y = 0$.

Let Ω be a simply connected domain lying in the half-plane $y > 0$, whose boundary Γ consists of a segment AB of the axis $y = 0$ and a smooth arc σ with endpoints at the points A and B.

Equations of the form (1) and (2) in the domain Ω, in the case $n = 1$, have by now been well studied ([1−7]). Attention was drawn to the importance of studying systems of the form (1) and (2), in particular, in the work ([1]).

In the present paper the first boundary-value problem for system (1) in the domain Ω is investigated. For system (2), as also in the case of a single equation ($n = 1$), the first boundary-value problem in this domain is not always correct. In the paper, for this system, conditions for the correctness of the first boundary-value problem and conditions for the correctness of the so-called problem E are established (see ([3])).

Denote by $\overset{\circ}{C}{}^2$ the set of twice continuously differentiable vectors that vanish in a boundary strip of the domain Ω. On this set we define a norm by the formula

$$\|u\|_+ = \int_\Omega \{y^m u_y^2 + u_x^2\}\, d\Omega.$$

The closure of the set $\overset{\circ}{C}{}^2$ in the plus-norm will be denoted by $W^1_{(2,m)}$. By $W^0_{(2,\mu)}$ we denote the set of all vectors f for which the integral

$$\int_\Omega \mu^{-1} f^2 \, d\Omega$$

is finite, where $\mu(x,y) > 0$ is a sufficiently smooth function in the domain Ω, with

$$\mu = \begin{cases} O(y^{m-2} |\ln y|^{-1-\varepsilon}), & \text{for } m \neq 1, \\ O(y^{-1} |\ln y|^{-2-\varepsilon}), & \text{for } m = 1. \end{cases}$$

We define a negative norm for each element $f \in W^0_{(2,\mu)}$ by the formula

$$\|f\|_- = \sup \frac{(f \cdot v)}{\|v\|_+}, \qquad v \in W^1_{(2,m)},$$

where

$$(f \cdot v) = \int_\Omega \{f_1 v_1 + \cdots + f_{nv} n\} \, d\Omega$$

is the scalar product in the pro-

space W^0_2. We denote by $W^{-1}_{(2,m)}$ the closure, with respect to the negative norm, of the set $W^0_{(2,\mu)}$.

Suppose that the matrices a, b, a_x, b_y, c satisfy the following conditions: a) in the case of system (1), for any $m > 0$, and for system (2), when $m \leqslant 1$ or $m \geqslant 2$, they are continuous in the closed domain $\overline{\Omega}$; b) in the case of system (2), when $1 < m < 2$, they are continuous in the closed domain $\overline{\Omega}_\delta$, where $\Omega_\delta = \Omega \cap (y = \delta > 0)$, and belong to $W^0_{(2,\mu)}$; c) a and b are symmetric; d) in any domain Ω' interior with respect to Ω, the matrices a, b, c have generalized derivatives up to the third order, square-summable, and the vector h has generalized derivatives up to the second order, square-summable; e) for system (1), for any $m > 0$, and for system (2), when $m \leqslant 1$, the inequality

$$c - \frac{1}{2} a_x - \frac{1}{2} b_y \geqslant 0,$$

holds, while for system (2), when $m > 1$, the inequality

$$c - \frac{1}{2} a_x - \frac{1}{2} \left[m E y^{m-1} + b \right]_y \geqslant 0,$$

holds, where E is the identity matrix; here the matrix inequalities are understood in the sense of inequalities for the corresponding quadratic forms constructed on arbitrary vectors of nonzero length.

Theorem 1. The system (1), for $h \in W_2^0$, has a unique solution u, twice continuously differentiable in the domain Ω, which vanishes on the boundary Γ in the mean, i.e.

$$\lim_{\delta_1 \to 0} \int_{\Gamma_{\delta_1}} u^2 \, d\Gamma_{\delta_1} = 0,$$

where δ_1 is the width of the boundary strip of the domain Ω.

Theorem 2. If $h \in W_2^0$ when $m \leqslant 1$ or $m \geqslant 2$, and $h \in W_{(2,m)}^{-1}$ when $1 < m < 2$, then, provided that in a neighborhood of $y = 0$ one of the following conditions is satisfied: 1) $m < 1$, b arbitrary; 2) $m = 1$, $b + E > 0$; 3) $m > 1$, $b > 0$, the system (2) has a unique solution u, twice continuously differentiable in the domain Ω, which vanishes on the boundary Γ of the domain Ω, where vanishing on Γ in the case $m < 1$ is understood in the mean, and for $m \geqslant 1$—in the mean on σ and in the weak sense on $\overline{AB}$, i.e.

$$\lim_{AB_\delta} \int u \cdot v \, dx = 0.$$

Theorem 3. If $h \in W_{(2,m)}^{-1}$, then, provided that in a neighborhood of $y = 0$ one of the following conditions is satisfied: 1) $m = 1$, $b + E < 0$; 2) $1 < m < 2$, $b + mEy^{m-1} < 0$; 3) $m \geqslant 2$, $b \leqslant 0$, there exists a unique solution u of system (2), twice continuously differentiable in the domain Ω, belonging to $W_{(2,m)}^1$ and vanishing only on σ.

We briefly outline the proof of Theorem 2 in the case $m > 1$. We shall call a vector u a weak solution of the first boundary-value problem for system (2), satisfying the boundary conditions in the weak sense, if the relation

$$(v \cdot h) = (L^* v \cdot u) \tag{3}$$

holds for all vectors v, twice continuously differentiable in the closed domain $\overline{\Omega}$, which vanish on the boundary strip σ. Here L^* denotes

$$L^* v = -(y^m v)_{yy} - v_{xx} - (av)_x - (bv)_y + vc.$$

It is easy to prove the validity of the inequalities

$$\|L^* v\|_- \geq \text{const} \, \|v\|_+, \qquad \text{const} > 0; \tag{4}$$

$$\|L^* v\|_{W_2^0} \geq \text{const} \, \|v\|_{W_2^0}, \qquad \text{const} > 0. \tag{5}$$

Lemma 1. *Every linear functional on the Hilbert space* $W^{-1}_{(2,m)}$ *can be represented in the form*

$$l(f) = (f \cdot v),$$

where $v \in W^{1}_{(2,m)}$, $f \in W^{-1}_{(2,m)}$.

To prove the existence of a weak solution of the first boundary-value problem, consider the linear functional

$$l(v) = (v \cdot h). \tag{6}$$

Using (4), we easily obtain:

$$l(v) = (v \cdot h) \leq \text{const}\, \|L^* v\|_{-}, \qquad \text{const} > 0.$$

In view of this inequality, we may regard expression (6) as a linear functional in $L^* v$ in the space $W^{-1}_{(2,m)}$. Extending this functional, by the Banach–Hahn theorem, to the whole space and using the general form of linear functionals in $W^{-1}_{(2,m)}$, we conclude that there exists an element $u \in W^{1}_{(2,m)}$ such that our functional on $W^{-1}_{(2,m)}$ can be represented in the form

$$(v \cdot h) = (u \cdot L^* v).$$

This is precisely the definition of a weak solution. Thus, for the case $1 < m < 2$ we have proved the existence of a weak solution of the first boundary-value problem. For sufficiently large m, the element $L^* v$ need no longer belong to the space $W^{-1}_{(2,m)}$, i.e., the arguments given above are not always applicable. Therefore, in the case $m \geq 2$, we proceed as follows: estimate expression (6), using (5):

$$l(v) = (v \cdot h) \leq \text{const}\, \|L^* v\|_{W_2^0}, \qquad \text{const} > 0.$$

Now we may regard (6) as a linear functional in $L^* v$ in the space W_2^0. Extending this linear functional to all of W_2^0 and using the Riesz theorem on the general form of a linear functional, we obtain the existence of a weak solution of the first boundary-value problem belonging to the space W_2^0; then we show that this solution also belongs to the space $W^{1}_{(2,m)}$.

Using a priori estimates for the operator $L^* \Delta^s$ (here Δ is the Laplace operator, s is a positive integer), by arguments analogous to those given above, we obtain the required differentiability of the weak solution.

Denote by $W_2^0(\Omega_{\delta_1})$ the set of vectors square-integrable in the domain Ω, equal to zero in the boundary strip of width greater than δ_1.

Lemma 2. *For every vector $f \in W_2^0(\Omega_{\delta_1})$ there exists a vector v, equal to zero in the δ_1-strip and belonging to the space $W^1_{(2,m)}$, for which the equality holds:*

$$(a \cdot f) = (v \cdot La),$$

where a are twice continuously differentiable vectors equal to zero in a strip of width greater than δ_1.

From this lemma and the theorem on differentiability of weak solutions it follows that every element $f \in W_2^0(\Omega_{\delta_1})$ can be approximated in the norm W_2^0 by elements of the form L^*v, where $v \in C_0^2$.

Lemma 3. The set of all vectors f belonging to the sum

$$\bigcup_{\delta>0} W_2^0(\Omega_\delta)$$

is dense in the space $W^{-1}_{(2,m)}$.

From this lemma there follows immediately

Lemma 4. Elements of the form L^*v, where $v \in C_0^2$, are dense in the space $W^{-1}_{(2,m)}$.

It is now easy to prove the uniqueness of the weak solution of the first boundary-value problem. Indeed, let the vector $u(x, y)$ be a weak solution, i.e., let equality (3) hold. We shall consider (3) only for elements $v \in C_0^2$. Applying Schwarz' s inequality and inequality (4) to the right-hand side, we obtain

$$(v \cdot h) \leq \text{const}\, \|L^*v\|_-, \qquad \text{const} > 0.$$

Since the vectors L^*v are dense in the space $W^{-1}_{(2,m)}$, applying Lemma 1 we obtain that there exists a unique element $\psi \in W^1_{(2,m)}$ such that

$$(v \cdot h) = (L^*v \cdot \psi).$$

The vector ψ and the weak solution u coincide, since the linear functional $(L^*v \cdot \psi - u)$ vanishes on a set dense in the space $W^{-1}_{(2,m)}$, i.e.,

$$\|\psi - u\|_+ = 0.$$

Thus, the uniqueness of the weak solution is proved.

The membership of the weak solution u in the domain Ω in the space $W^1_{(2,m)}$ and the differentiability theorem make it possible to conclude that u satisfies the stated boundary conditions.

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Institute of Mathematics with Computing Center
Siberian Branch of the Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.