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Abstract

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CRITERIA FOR ONE-SIDED INVERTIBILITY OF ELEMENTS OF NORMED RINGS AND THEIR APPLICATIONS

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In the present communication a new general ring-theoretic approach is proposed for the solution and investigation of certain classes of equations, including one-dimensional singular integral equations, systems of equations of the form

$$\sum_{j=1}^{\infty} \alpha_{k-j}\xi_j = \eta_k \qquad (k = 1, 2, ...) \tag{1}$$

and those close to them. It turns out that the main propositions, for example concerning the system (1), established by M. G. Krein ([1]), are in essence criteria for one-sided invertibility of certain classes of elements of a normed ring. This approach has made it possible to obtain new general criteria for one-sided invertibility of elements, from which, in particular, generalizations of the theorems of M. G. Krein ([1]) on the system (1) and generalizations of the principal theorems on other classes of equations, in a certain sense close to the system of equations (1), follow.

§ 1.1. In what follows R denotes an arbitrary normed ring with unit e.

Let $v \in R$ be an element having an inverse $v^{(-1)}$ ($\in R$) strictly on the left: $v^{(-1)}v = e,\ vv^{(-1)} \neq e$. In what follows we shall assume that $|v| = |v^{(-1)}| = 1$*. Introduce the following notation: $K_{(v)}$ is the linear hull of all elements $v^{(j)}$ ($j = 0, \pm1, ...$), where $v^{(j)} = v^j$ ($j = 0, 1, ...$) and $v^{(j)} = (v^{(-1)})^{-j}$ ($j = -1, -2, ...$); $K^+_{(v)}$, $K^-_{(v)}$ are respectively the linear hulls of $v^{(j)}$ ($j = 0, -1, ...$) and $v^{(j)}$ ($j = 0, -1, ...$); $R_{(v)}$, $R^+_{(v)}$, and $R^-_{(v)}$ are the closures (in the norm of the ring R) respectively of $K_{(v)}$, $K^+_{(v)}$, and $K^-_{(v)}$.

It is obvious that $R^{\pm}_{(v)}$ are commutative normed subrings of R, while $R_{(v)}$ is a subspace of R. To each element $r = \sum \alpha_j v^{(j)} \in K_{(v)}$ we associate the function $r(\zeta) = \sum \alpha_j \zeta^j$ ($|\zeta| = 1$). On the linear space $K_{(v)}$ define a new multiplication by putting $r_1 \circ r_2 = r$, where the element $r \in K_{(v)}$ is determined by the equality $r(\zeta) = r_1(\zeta) r_2(\zeta)$. It is easily proved that for any pair of elements $r_1, r_2 \in K_{(v)}$:

$|r_1 \circ r_2| \leq |r_1|\,|r_2|$. Hence it follows that the new multiplication operation can be extended by continuity to all pairs of elements of $R_{(v)}$. With this definition of multiplication, $R_{(v)}$ becomes a normed commutative ring with two mutually inverse generators v and $v^{(-1)}$. The bicompactum of maximal ideals of the ring $R_{(v)}$ is homeomorphic to the entire unit circle. By $a(\zeta)$ ($|\zeta| = 1$) we denote the function on the bicompactum of maximal ideals of the ring $R_{(v)}$ corresponding to the element $a \in R_{(v)}$.

Theorem 1. In order that an element $a \in R_{(v)}$ be invertible in R at least on one side, it is necessary and sufficient that the condition

$$a(\zeta) \neq 0 \qquad (|\zeta| = 1). \tag{2}$$

be satisfied.

* The condition $|v| = |v^{(-1)}| = 1$ may be replaced by the condition $\sup |v^{(j)}| < \infty$.

If condition (2) is satisfied, then for $\varkappa(a) > 0$, where

$$\varkappa(a) = \frac{1}{2\pi}\,[\arg a(e^{i\varphi})]_{\varphi=0}^{\varphi=2\pi},$$

the element a is invertible in R strictly from the left; for $\varkappa(a) < 0$ it is invertible in R strictly from the right; and for $\varkappa(a) = 0$ the element a has in R a two-sided inverse.

In the proof of this theorem an effective construction is given of the element $a^{(-1)}$, inverse from the corresponding side to the element a. The formula expressing the relation between the elements $a^{(-1)}$ and a is considerably simplified if every element $y \in R_{(v)}$ can be represented in the form

$$y = y_+ + y_- \quad (y_+ \in R^+_{(v)},\ y_- \in R^-_{(v)}). \tag{3}$$

Namely, the following holds.

Theorem 2. In order that every element $a \in R_{(v)}$ satisfying condition (2) admit a factorization, i.e. admit a representation in the form

$$a = a_-\, v^{(\varkappa)}\, a_+ \quad (\varkappa = \varkappa(a)),$$

where $a_\pm$ are invertible elements respectively of the rings $R^\pm_{(v)}$, it is necessary and sufficient that every element $y \in R_{(v)}$ be representable in the form (3).

If condition (3) is fulfilled, then the element $a^{(-1)}$ is given by the formula

$$a^{(-1)} = a_+^{-1}\, v^{(-\varkappa)}\, a_-^{-1}.$$

This theorem is a generalization of M. G. Krein's theorem ([1]) on the factorization of functions on the unit circle; its proof is based on a theorem of G. E. Shilov ([2]), §13.

§ 1,2. Let $\mathfrak{B}$ be a Banach space, $\mathfrak{R}$ the ring of all linear bounded operators acting in $\mathfrak{B}$, and let V be an isometric operator from $\mathfrak{R}$, possessing a left inverse $V^{(-1)}$ such that $|V^{(-1)}| = 1$.* It is clear that Theorems 1 and 2 are applicable to the elements of the ring $\mathfrak{R}_{(V)}$. We note, incidentally, that if $\mathfrak{B} = \mathfrak{h}$ is a Hilbert space, then for every $A \in R_{(V)}$

$$|A| = \max_{|\zeta|=1} |A(\zeta)|,$$

and, consequently, in this case the set of all functions $A(\zeta)$ ($|\zeta| = 1$; $A \in \mathfrak{R}_{(V)}$) consists of all continuous functions on the unit circle.

For the ring $\mathfrak{R}_{(V)}$ in the general case of a Banach space $\mathfrak{B}$, Theorems 1 and 2 admit the following refinement:

Theorem 3. In order that an operator $A \in \mathfrak{R}_{(V)}$ be normally solvable and that at least one of the equations $Ax = 0$, $A^*f = 0$ ($x \in \mathfrak{B}$; $f \in \mathfrak{B}^*$) have a finite number of linearly independent solutions, it is necessary that the condition

$$A(\zeta) \neq 0 \quad (|\zeta| = 1) \tag{4}$$

be satisfied.

If condition (4) is satisfied and $\varkappa(A) < 0$, then the subspace of all solutions $\mathfrak{Z}$ of the equation $Ax = 0$ has dimension $\alpha|\varkappa(A)|$, where α ($\leq \infty$) is the dimension of the subspace of all solutions of the equation $Vx = 0$. If $\alpha < \infty$, then there exists an α-dimensional subspace Ω such that

$$\mathfrak{Z} = \Omega \dotplus V\Omega \dotplus \cdots \dotplus V^{m-1}\Omega \quad (m = |\varkappa(A)|).$$

The results listed above, with some additions, were obtained earlier by M. G. Krein ([1]) for the following case: $\mathfrak{B}$ is one of the spaces of sequences $\xi = \{\xi_j\}_1^\infty$ of complex numbers (l_p) $(1 \leq p < \infty)$, (m),

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* In order that an isometric operator V have a left inverse $V^{(-1)}$ such that $|V^{(-1)}| = 1$, it is necessary and sufficient that there exist in $\mathfrak{R}$ a projector P with norm equal to one for which $P\mathfrak{B} = V\mathfrak{B}$.

$(c), (c_0)$; V is the shift operator: $V\{\xi_1, \xi_2, ...\} = \{0, \xi_1, \xi_2, ...\}$, and $V^{(-1)}$ is defined by the equality $V\{\xi_j\}_1^\infty = \{\xi_{j+1}\}_1^\infty$. In addition, in (1) it was also assumed that the operator $A \in \mathfrak{R}_{(V)}$ has the form*

$$A = \sum_{j=-\infty}^{\infty} \alpha_j V^{(j)} \qquad \left(\sum_j |\alpha_j| < \infty \right). \tag{5}$$

With the aid of Theorems 1-3 one can show that in this case the entire theory constructed by M. G. Krein ([1]) remains valid if the range space $\mathfrak{B}$ is replaced by any Banach space $\mathfrak{B}'$ of sequences $\xi = \{\xi_j\}_1^\infty$ having the property: if $\xi = \{\xi_j\}_1^\infty \in \mathfrak{B}'$, then the vectors $\xi' = \{0, \xi_1, \ldots\}$ and $\xi'' = \{\xi_{j+1}\}_1^\infty$ also belong to $\mathfrak{B}'$, and moreover $|\xi'| \le |\xi|$ and $|\xi''| \le |\xi|$.

§ 2.1. Let x be an arbitrary element of R and let Λ_j^+ $(j = 0, 1, \ldots, \omega^+;\ \omega^+ \le \infty)$, Λ_j^- $(j = 1, 2, \ldots, \omega^-;\ \omega^- \le \infty)$ be certain distinct connected components of the regularity domain of the element x, with $\lambda = \infty \in \Lambda_0^+$.

Introduce the following notation: $\mathfrak{K}^+$ is the linear span of the system x^n, $(x - \lambda_j^+ e)^{-n}$ $(\lambda_j^+ \in \Lambda_j^+;\ j = 1, 2, \ldots, \omega^+;\ n = 0, 1, \ldots)$; $\mathfrak{K}^-$ is the linear span of the system $(x - \lambda_j^- e)^{-n}$ $(\lambda_j^- \in \Lambda_j^-;\ j = 1, 2, \ldots, \omega^-;\ n = 0, 1, \ldots)$; $\mathfrak{K}$ is the linear span of the union of both systems; $\mathfrak{R}, \mathfrak{R}^+, \mathfrak{R}^-$ are the closures (in the norm of the ring R) respectively of the rings $\mathfrak{K}, \mathfrak{K}^+$, and $\mathfrak{K}^-$; $\Lambda^\pm = \bigcup_j \Lambda_j^\pm$ and S is the complement of the domain $\Lambda^+ \cup \Lambda^-$.

Obviously, $\mathfrak{R}, \mathfrak{R}^\pm$ are normed subrings of the ring R, and these subrings do not depend on the choice of the numbers $\lambda_j^\pm$ in the corresponding components $\Lambda_j^\pm$. The bicompacts of maximal ideals of the rings $\mathfrak{R}, \mathfrak{R}^+, \mathfrak{R}^-$ are homeomorphic respectively to the sets $S, S \cup \Lambda^-$, $S \cup \Lambda^+$. As before, by $z(\xi)$ we shall denote the function of the element z on the bicompact of maximal ideals of the ring.

If the set S coincides with the spectrum of the element x in R, then the spectrum of every element $z \in \mathfrak{R}$ is the same in the rings R and $\mathfrak{R}$.

Denote by $\varkappa_j^\pm(r)$ $(j = 1, 2, \ldots, \omega_j^\pm)$ the functional defined on all invertible elements $r \in \mathfrak{K}$ $(r(\xi) \ne 0;\ \xi \in S)$ and equal to the difference between the number of poles and the number of zeros in $\Lambda_j^\pm$ of the function $r(\xi)$. The functionals $\varkappa_j^\pm(r)$ extend by continuity to the set O of all invertible elements of $\mathfrak{R}$ and have the following properties: 1) $\varkappa_j^\pm(a)$ is an integer for every $a \in O$; 2) $\varkappa_j^\pm(a_1 a_2) = \varkappa_j^\pm(a_1) + \varkappa_j^\pm(a_2)$ $(a_1, a_2 \in O)$.

Theorem 4. The general form of every functional $\varkappa(a)$, defined on O and possessing properties 1) and 2), is given by the formula

$$\varkappa(a) = \sum_j c_j^+ \varkappa_j^+(a) + \sum_j c_j^- \varkappa_j^-(a) \quad (a \in O), \tag{6}$$

where $c_j^\pm$ are arbitrary integers.

The sums on the right-hand side of equality (6) are meaningful, since for each $a \in O$ only a finite number of the functionals $\varkappa_j^\pm$ may be nonzero.

Theorem 5. In order that every element $a \in \mathfrak{R}$ satisfying the condition $a(\xi) \neq 0$ $(\xi \in S)$ be representable in the form

$$a = a_{-}(x - \lambda^{-}e)^{\varkappa} a_{+} \qquad \left(\lambda^{-} \in \Lambda^{-}; \quad \varkappa = \varkappa(a) = \sum_{j} \varkappa_{j}^{-}(a)\right), \tag{7}$$

* An operator generated in one of the indicated spaces B by a matrix $\|a_{j-k}\|$, $j, k = 1, 2, \ldots$, satisfying the condition $\sum |\alpha_j| < \infty$, can obviously be represented in the form (5).

where $a_{\pm}$ are invertible elements, respectively, of the rings $\mathcal{R}^{\pm}$; it is necessary and sufficient that every element $z \in \mathcal{R}$ be representable in the form

$$z = z_{+} + z_{-} \qquad (z_{+} \in \mathcal{R}^{+},\ z_{-} \in \mathcal{R}^{-}). \tag{8}$$

This theorem is a generalization of Theorem 2 and of some propositions from the theory of the Hilbert boundary-value problem [3]. Its proof is based on a theorem of G. E. Shilov ([2], §13).

§ 2.2. Suppose that in the ring R there exists an idempotent element p $(p^2 = p;\ p \neq e)$ with respect to which x has the following properties:

$$pxp = xp, \qquad p(x - \lambda_j^{+}e)^{-1}p = (x - \lambda_j^{+}e)^{-1}p \quad (j = 1, 2, \ldots, \omega^{+});$$

$$qxq \neq xq, \qquad q(x - \lambda_j^{-}e)^{-1}q = (x - \lambda_j^{-}e)^{-1}q \quad (q = e - p;\ j = 1, 2, \ldots, \omega^{-}).$$

Under these assumptions the following holds.

Theorem 6. Let $b, c \in \mathcal{R}$. If

$$b(\zeta) \neq 0, \qquad c(\zeta) \neq 0 \qquad (\zeta \in S), \tag{9}$$

then for $\varkappa(a) > 0$ $\left(a = bc^{-1},\ \varkappa(a) = \sum_j \varkappa_j^{-}(a)\right)$ the element $bp + cq$ $(pb + qc)$ is strictly left-invertible in R; for $\varkappa(a) < 0$ it is strictly right-invertible in R; and for $\varkappa(a) = 0$ the element $bp + cq$ $(pb + qc)$ has a two-sided inverse in R. If condition (8) is satisfied in the ring $\mathcal{R}$, then in all three cases

$$(bp + cq)^{(-1)} = (a_{+}p + a_{-}^{-1}q)\left[(x - \lambda^{-}e)^{-\varkappa(a)}p + q\right]c^{-1}a_{-}^{-1} \qquad (\lambda^{-} \in \Lambda^{-}),$$

$$(pb + qc)^{(-1)} = c^{-1}a_{+}^{-1}\left[p(x - \lambda^{-}e)^{-\varkappa(a)} + q\right](pa_{-}^{-1} + qa_{+}) \qquad (\lambda^{-} \in \Lambda^{-}),$$

where $a_\pm$ are determined from equality (7), if one sets $a = bc^{-1}$. If at no point $\lambda \in S$ are the elements $(x - \lambda e)p + q$, $\big[(x - \lambda^{-} e)^{-1} - (\lambda - \lambda^{-})^{-1} e\big] q + p$ right-invertible in R, and the elements $q(x - \lambda e) + p$, $p\big[(x - \lambda^{-} e)^{-1} - (\lambda - \lambda^{-})^{-1} e\big] + q$ have no left inverses in R, then (9) is a necessary condition for the element $bp + cq$ $(pb + qc)$ to be invertible in R from at least one side.

For lack of space we omit here the statement of the theorem analogous to Theorem 3.

§ 2.3. As a consequence of the results of §§ 2.1 and 2.2 one can obtain a generalization of the theorems for paired systems with Toeplitz matrices of type (1) and their transposes ([4], §7). In addition, all the fundamental assertions on singular integral equations with discontinuous coefficients on multiply connected closed contours in L_p $(1 < p < \infty)$ [5–8] follow from these results.

The results presented in this communication are generalized to the case of matrices composed of elements of the classes considered.

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Note: Figure translations are in progress. See original paper for figures.