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THEORY OF ELASTICITY

1962

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Abstract

Full Text

THEORY OF ELASTICITY

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ON A NONLINEAR BOUNDARY-VALUE PROBLEM OF THE THEORY OF ANALYTIC FUNCTIONS OCCURRING IN CERTAIN ELASTIC-PLASTIC PROBLEMS

(Presented by Academician Yu. N. Rabotnov on 12 VI 1962)

§ 1. Boundary-value problem.

1°. Suppose it is required to determine a function $\omega(z)$, analytic in the upper half-plane $\operatorname{Im} z > 0$, from the nonlinear boundary condition on the real axis t

$$|\omega(t)| = \alpha(t) \quad (t \in L); \qquad \operatorname{Re}[(a(t) - ib(t))\omega(t)] = 0 \quad (t \in M). \tag{1}$$

Here $a(t)$, $b(t)$, and $\alpha(t)$ are functions continuous almost everywhere, satisfying a Hölder condition on intervals of continuity and at the infinitely distant point $(a + ib \neq 0)$; $L = L_1 + \dots + L_n$; L_k is the segment $-\infty < a_k \le t \le b_k < \infty$; M is the set of points of the real axis lying outside L.

We shall require, at least, integrability of the function $\omega(z)$ at the ends of the segments $t = a_k$ and $t = b_k$, as well as at the points of discontinuity of the coefficient $a - ib$ $(t = c_k)$ and of the function $\alpha(t)$ $(t = d_k)$. We indicate a method for solving the boundary-value problem (1), based on reducing it to a nonlinear Riemann boundary-value problem, solvable in closed form by methods analogous to the classical methods for solving linear boundary-value problems developed in the monographs of N. I. Muskhelishvili ([1]) and F. D. Gakhov ([2]).

2°. By the canonical function of the nonlinear boundary-value problem (1) we shall mean the piecewise-analytic function $X(z)$, which is the canonical function of the Riemann problem

$$X^+(t) = G(t)X^-(t); \qquad G(t) = \begin{cases} -\dfrac{a(t) + ib(t)}{a(t) - ib(t)} & (t \in M), \\ 1 & (t \in L), \end{cases} \tag{2}$$

where near the points $t = c_k$ the class of $X(z)$ coincides with the prescribed class of the function $\omega(z)$, and at the points a_k and b_k the function $X(z)$ is bounded.

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The canonical function of problem (1) defined in this way is written in the form ([1,2])

$$X(z)=\prod_{k=1}^{n}(z-b_k)^{-\varkappa_k}e^{\Gamma(z)},\qquad \Gamma(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln G(\tau)}{\tau-z}\,d\tau. \tag{3}$$

Here $\varkappa=\sum\varkappa_k$ is the index of the Riemann problem (2), determined in the usual way (([2]), p. 436).

Introduce the function $\Phi(z)$, analytic in the entire z-plane except, possibly, the real axis,

$$\Phi(z)=\begin{cases}\omega(z)/X^{+}(z), & \text{for } \operatorname{Im}z>0,\\ \overline{\omega}(z)/X^{-}(z), & \text{for } \operatorname{Im}z<0.\end{cases} \tag{4}$$

The boundary condition (1), with the aid of formulas (4) and (2), can be written in the form

$$\Phi^{+}\Phi^{-}=\alpha^2X^{-2}\quad(t\in L);\qquad \Phi^{+}=\Phi^{-}\quad(t\in M). \tag{5}$$

The function $\Phi(z)$ has zero order at the points c_k and order $\varkappa$ at infinity. Thus, problem (1) has been reduced to the nonlinear Riemann boundary-value problem: determine a function $\Phi(z)$, analytic outside the cuts L, having finite order at infinity, under the condition

$$\Phi^{+}\Phi^{-}=\beta(t)\quad(t\in L),\qquad \text{where } \beta(t)=\alpha^2(t)X^{-2}(t). \tag{6}$$

Problem (6) can be reduced to a linear Riemann boundary-value problem, generally speaking, for a multivalued analytic function with logarithmic singularities.

3°. First suppose that the simple smooth closed contour L separates the interior domain D^{+} and the exterior domain D^{-}, and that $\beta(t)$ does not vanish or become infinite on the contour L. Let the function $\Phi^{+}(z)$ have m zeros in D^{+} at the points $z=\mu_i$, and $\Phi^{-}(z)$ have n zeros in D^{-} at the points $z=\lambda_i$ ($m-n=\varkappa$, where $\varkappa=\operatorname{Ind}\beta(t)$). For definiteness we assume that the origin lies in the domain D^{+}. Then one can show that the solution of problem (6) is determined up to an arbitrary constant C by the following formulas:

$$\Phi^{+}(z)=C\prod_{i=1}^{m}(z-\mu_i)\prod_{i=1}^{n}(z-\lambda_i)^{-1}e^{\Gamma^{+}(z)},$$

$$\Phi^{-}(z)=C^{-1}z^{\varkappa}\prod_{i=1}^{m}(z-\mu_i)^{-1}\prod_{i=1}^{n}(z-\lambda_i)e^{-\Gamma^{-}(z)}, \tag{7}$$

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$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln[\beta(\tau)\tau^{-\varkappa}]}{\tau - z}\, d\tau.$$

If no additional requirements are imposed on $\Phi^+(z)$ and $\Phi^-(z)$ concerning the number and location of zeros, then the boundary-value problem (6) will have an infinite number of solutions, determined by formulas (7) for arbitrary numbers λ_i, μ_i, m, n such that $\mu_i \in D^+$, $\lambda_i \in D^-$, $m - n = \varkappa$. This is the most essential distinction between the boundary-value problem (6) and the linear Riemann boundary-value problem ([1,2]).

4°. Now suppose that the open smooth contour L consists of n arcs with endpoints a_k and b_k, while $\beta(t)$, as before, does not vanish or become infinite on the contour L.

One can show that the solution of the boundary-value problem (6) has, at the ends of the cuts a_k and b_k, either a nonintegrable power singularity or is bounded, so that from the requirement of integrability of the solution at the ends of the cuts a_k and b_k its boundedness in a neighborhood of these endpoints follows.

Let an analytic solution of problem (6) have m zeros at the points $z = \nu_i$. For definiteness, suppose that the point $z = 0$ belongs to the contour L. Then one can show that the solution of problem (6) is determined by the following formulas:

$$\Phi(z) = z^{-m}\prod_{i=1}^{m}(z - \nu_i)e^{\Gamma(z)},$$

$$\Gamma(z) = \frac{X_n(z)}{2\pi i}\int_L \ln\left[\frac{\beta(\tau)\tau^{2m}}{\sum_{i=1}^{m}(\tau - \nu_i)^2}\right]\frac{d\tau}{X_n^+(\tau)(\tau - z)}, \tag{8}$$

where the $(n-1)$ conditions must be satisfied

$$\int_L \frac{\tau^k}{X_n^+(\tau)}\ln\left[\frac{\beta(\tau)\tau^{2m}}{\sum_{i=1}^{m}(\tau - \nu_i)^2}\right]d\tau = 0, \qquad k = 0, 1, \dots, n-2.$$

Here

$$X_n(z) = \prod_{i=1}^{n}(z - a_i)^{1/2}(z - b_i)^{1/2}.$$

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Thus, analogously to the case of a closed contour, for the complete determination of the problem it is, generally speaking, necessary to prescribe the number and location of the zeros of the required solution; otherwise, if there is no such prescription, the number of solutions is infinite and the solutions are determined by formulas (8) with arbitrary m, C, v_i. When the number of zeros m is prescribed, the solution is determined up to m arbitrary constants.

At points of discontinuity of the first kind of the coefficient $\beta(t)$, the solution of problem (6) has a power singularity.

Remark. In an analogous way one can obtain the solution of a nonlinear boundary-value problem of Riemann-problem type

$$[\Phi^+(t)]^n = G(t)\Phi^-(t) + g(t),$$

where n is an arbitrary integer, for a closed contour, and of the problem

$$[\Phi^+(t)]^n = G(t)\Phi^-(t)$$

for an open contour; moreover, for $n = 0$ the solution of the problem, generally speaking, does not exist; for $n > 0$ the principal characteristic of the problem is the index of $G(t)$, and for $n < 0$ it is the number of zeros of the function $\Phi(z)$. Difficulties arise in the general case in determining the character of the product of the complex constants entering the solution (see [3], where the first problem is considered for a closed contour with integer $n > 1$).

§ 2. An elastic-plastic problem under conditions of antiplane deformation

By antiplane deformation is meant the state of stress in a cylindrical body of infinitely great height, arising under the action of loads directed along the generators of the cylinder and constant along the generators. Let the contour of the body be formed by segments of straight and curved lines, the straight-line segments being free of load, while the segments of curved arcs, arbitrarily loaded, are entirely encompassed by the plastic zone. It is required to determine the stress vector

$$\vec{\tau} = \tau_{xz} + i\tau_{yz}$$

and the displacement w, as well as the unknown boundary separating the elastic and plastic regions. (The relation between stresses and deformations is described by the Prandtl diagram.) On the boundary of the elastic and plastic zones the vector $\vec{\tau}$ is continuous. In the plastic region there is a first-order hyperbolic equation whose characteristics are straight lines, the vector $\vec{\tau}$ being orthogonal at every point to a characteristic ($|\vec{\tau}| = k$ everywhere in the plastic region, where k is the plasticity constant), and the displacement $w = \text{const}$ along a characteristic. In the elastic region the basic representations hold [4]:

$$\vec{\tau} = \mu f'(z),$$

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$$w = \operatorname{Re} f(z),$$

where $f(z)$ is a function analytic in the elastic region. Passing to the upper half-plane $\operatorname{Im}\zeta > 0$ by means of the conformal transformation $z = \omega(\zeta)$ (we denote $\mu f'[\omega(\zeta)] = kF(\zeta)$), in the ζ-plane one obtains boundary-value problem (1) for the function $F(\zeta)$. Indeed, on the portions of the real axis that are the image of the unknown part of the boundary, one has $|F(\zeta)| = 1$, while on the remaining portions the condition of absence of load is

$$\operatorname{Re}\big[(\operatorname{tg}\theta_j - i)F(\zeta)\big] = 0,$$

where θ_j is the angle formed by the j-th straight line of the contour of the body with the x-axis. After finding the function $F(\zeta)$ and substituting it into the boundary condition for $\omega(\zeta)$, in order to determine $\omega(\zeta)$ one obtains the well-studied Hilbert problem for the half-plane [1,2].

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Received
31 V 1962

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Note: Figure translations are in progress. See original paper for figures.

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