MATHEMATICS

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Abstract

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SOME LINEAR PROCESSES OF SUMMATION OF FOURIER SERIES AND BEST APPROXIMATION

(Presented by Academician V. I. Smirnov, March 15, 1962)

1. Consider the space L_p $(1 \leq p \leq \infty)$ of all measurable periodic functions of period 2π, for which, when $1 \leq p < \infty$,

$$\|f(x)\|_{L_p} = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p} < \infty,$$

and, when $p = \infty$,

$$\|f(x)\|_{L_\infty} = \operatorname{vrai\,sup}_{0 \leq x \leq 2\pi} |f(x)| < \infty.$$

Let $\{\lambda_k^{(n)}\}$ $(k = 0, 1, 2, \dots, n;\ n = 1, 2, \dots;\ \lambda_0^{(n)} = 1;\ \lambda_{n+1}^{(n)} = 0)$ be an arbitrary triangular matrix of numbers, and

$$U_n(f; x; \lambda) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx),$$

where a_k, b_k are the Fourier coefficients of the function $f(x)$. Denote

$$R_n(f; \lambda)_{L_p} = \|f(x) - U_n(f; x; \lambda)\|_{L_p}.$$

Theorem 1. If $f(x) \in L_p$ $(1 \leq p \leq \infty)$, then for any matrix $\{\lambda_k^{(n)}\}$ the inequality

$$R_n(f; \lambda)_{L_p} \leq C \left\{ \sum_{k=0}^{n} \left| \lambda_k^{(n)} - 2\lambda_{k+1}^{(n)} + \lambda_{k+2}^{(n)} \right| (n - k + 1) E_k(f)_{L_p} \times \right.$$
$$\left. \times \sum_{\nu=n-k}^{n} \frac{1}{\nu + 1} + \left| 1 - \lambda_1^{(n)} \right| \sum_{\nu=0}^{n} E_\nu(f)_{L_p} \right\}, \qquad (1)$$

where

$$E_n(f)_{L_p} = \inf_{\alpha_k, \beta_k} \left\| f(x) - \sum_{k=0}^{n} (\alpha_k \cos kx + \beta_k \sin kx) \right\|_{L_p} \qquad (1 \leq p \leq \infty),$$

and C is an absolute constant. Moreover, in the case $1 < p < \infty$,

$$R_n(f; \lambda)_{L_p} \leq C_p \sum_{k=0}^{n} \left| \lambda_k^{(n)} - \lambda_{k+1}^{(n)} \right| E_k(f)_{L_p}. \tag{2}$$

Inequality (1) for $p = \infty$ and $p = 1$ may be regarded as a generalization of Lebesgue' s inequality

$$\|f(x) - S_n(f; x)\|_{L_p} \leq C E_n(f)_{L_p} \ln n \qquad (p = 1,\ p = \infty;\ n > 1),$$

which corresponds to the case $\lambda_k^{(n)} = 1$ $(k = 0, 1, \dots, n)$, while inequality (2) is a generalization of the well-known inequality

$$\|f(x) - S_n(f; x)\|_{L_p} \leq C_p E_n(f)_{L_p} \qquad (1 < p < \infty),$$

where $S_n(f; x)$ is the partial sum of the Fourier series of the function $f(x)$.

2. Inequality (1) contains various estimates for particular methods of summation of Fourier series. We shall consider some of them here.

Normal Zygmund means. Let

$$\lambda_k^{(n)} = 1 - \left(\frac{k}{n+1} \right)^r \quad (k = 0, 1, 2, \dots, n;\ r \geqslant 1). \tag{3}$$

Theorem 2. For the matrix (3), for any $r \geqslant 1$, the inequality

$$R_n(f; \lambda)_{L_p} \leqslant \frac{C}{n^r} \sum_{\nu=1}^{n} \nu^{r-1} E_{\nu-1}(f)_{L_p} \quad (1 \leqslant p \leqslant \infty), \tag{4}$$

holds, where C is an absolute constant*.

In particular, for $r = 1$, for Fejér sums the inequality

$$R_n(f; \lambda)_{L_p} \leqslant \frac{C}{n} \sum_{k=0}^{n} E_k(f)_{L_p} \quad (1 \leqslant p \leqslant \infty). \tag{5}$$

holds.

The last estimate can be obtained directly from the integral representation for Fejér sums[**]. Indeed, by monotonicity of

$$\omega_2(f;t)_{L_p} = \sup_{|h|\le t} \|f(x+h) - 2f(x) + f(x-h)\|_{L_p},$$

$$\left\| f(x) - \frac{1}{n}\sum_{\nu=0}^{n-1} S_\nu(f;x) \right\|_{L_p} \leqslant C_1 \left\{ \omega_2\left(f;\frac{1}{n}\right)_{L_p} + \frac{1}{n}\int_{\pi/n}^{\pi} \frac{\omega_2(f;t)_{L_p}}{t^2}\,dt \right\} \leqslant$$

$$\leqslant C_2 \frac{1}{n}\int_{\pi/n}^{\pi} \frac{\omega_2(f;t)_{L_p}}{t^2}\,dt = \frac{C_2}{n}\sum_{\nu=1}^{n-1}\int_{\pi/(\nu+1)}^{\pi/\nu} \frac{\omega_2(f;t)_{L_p}}{t^2}\,dt.$$

It remains to take into account that (see (3), p. 344)

$$\omega_2\left(f;\frac{1}{n}\right)_{L_p} \leqslant \frac{M}{n^2}\sum_{k=1}^{n} kE_{k-1}(f)_{L_p} \quad (1 \leqslant p \leqslant \infty).$$

Bernstein-Rogozinskii sums and Jackson-Vallée-Poussin sums. Let

$$\lambda_k^{(n)} = \cos\frac{k\pi}{2n+1} \quad (k = 0, 1, \dots, n); \tag{6}$$

$$\lambda_k^{(2n)} = \begin{cases} 1 - \dfrac{3}{2}\left(\dfrac{k}{n}\right)^2 + \dfrac{3}{4}\left(\dfrac{k}{n}\right)^3, & (0 \leqslant k \leqslant n), \\ \dfrac{1}{4}\left(2 - \dfrac{k}{n}\right)^3, & (n \leqslant k \leqslant 2n), \\ 0, & (k > 2n). \end{cases} \tag{7}$$

Theorem 3. For the matrices (6) and (7) the inequality

$$R_n(f;\lambda)_{L_p} \leqslant \frac{C}{n^2}\sum_{\nu=1}^{n} \nu E_{\nu-1}(f)_{L_p} \quad (1 \leqslant p \leqslant \infty), \tag{8}$$

holds, where C is an absolute constant.

[*] Theorem 2 was reported by the author on 12 X 1961 at the seminar of the Department of Higher Mathematics of the Dnepropetrovsk Agricultural Institute. We note that the means appearing on the right-hand side of inequality (4) play an important role in inverse theorems of constructive function theory (see (3),

Chs. 6 and 7) and, as far as we know, were first introduced in paper (1) (see (1), Theorem 4).

** Estimate (5) was obtained by another method by S. B. Stechkin (5).

3. There exist examples showing that, in the general case, for $p = \infty$ and $p = 1$ the estimates given in Theorems 2 and 3 cannot be improved in order.

At the same time, using the methods applied by the author earlier in papers $(^{2,4})$, one can show that in a number of cases these estimates are crude even in order, if $1 < p < \infty$. In particular, if $f(x) \in L_p$, $(1 < p < \infty)$, and $E_n(f)_{L_p} = O\left(\frac{1}{n}\right)$, then, for example, for Fejér sums we have the inequality

$$\left\| f(x) - \frac{1}{n}\sum_{\nu=0}^{n-1} S_\nu(f;x) \right\|_{L_p} \le C_p \begin{cases} \frac{1}{n}(\ln n)^{1/p}, & (1 < p \le 2), \\ \frac{1}{n}(\ln n)^{1/2}, & (2 \le p < \infty), \end{cases}$$

which gives a sharper estimate than the known $O(\ln n/n)$.

Theorem 4. If $f(x) \in L_p$, $(1 < p < \infty)$, then for the matrix (3) the inequality

$$R_n(f;\lambda)_{L_p} \le \frac{C_p}{n^r}\left\{\sum_{\nu=1}^{n} \nu^{\gamma r-1} E^{\gamma}_{\nu-1}(f)_{L_p}\right\}^{1/\gamma}, \tag{9}$$

holds, where $\gamma = p$ for $1 < p \le 2$, and $\gamma = 2$ for $2 \le p < \infty$.

Theorem 5. If $f(x) \in L_p$, $(1 < p < \infty)$, then for the matrices (6) and (7) the inequality

$$R_n(f;\lambda)_{L_p} \le \frac{C_p}{n^2}\left\{\sum_{\nu=1}^{n} \nu^{2\gamma-1} E^{\gamma}_{\nu-1}(f)_{L_p}\right\}^{1/\gamma}, \tag{10}$$

holds, where $\gamma = p$ for $1 < p \le 2$, and $\gamma = 2$ for $2 \le p < \infty$.

There are examples showing that, in this form, these inequalities, in order, can no longer be improved in the general case.

4. The same estimates as in the theorems stated above* are valid if, in the left-hand sides of the corresponding inequalities, instead of the sums $U_n(f;x;\lambda)$, one considers the interpolation sums

$$U^*_n(f;x;\lambda) = \frac{a_0^{(n)}}{2} + \sum_{\nu=1}^{n} \lambda_\nu^{(n)} \left(a_\nu^{(n)} \cos \nu x + b_\nu^{(n)} \sin \nu x\right),$$

where

$$a_\nu^{(n)} = \frac{2}{2n+1} \sum_{k=0}^{2n} f\left(\frac{2k\pi}{2n+1}\right) \cos \nu \frac{2k\pi}{2n+1},$$

$$b_\nu^{(n)} = \frac{2}{2n+1} \sum_{k=0}^{2n} f\left(\frac{2k\pi}{2n+1}\right) \sin \nu \frac{2k\pi}{2n+1}.$$

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CITED LITERATURE

1. A. F. Timan, M. F. Timan, DAN, 71, No. 1 (1950).
2. M. F. Timan, Matem. sborn., 46 (88), No. 1, 125 (1958).
3. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.
4. M. F. Timan, Izv. Vyssh. ucheb. zaved., ser. matem., No. 6, 109 (1961).
5. S. B. Stechkin, Tr. Matem. inst. im. V. A. Steklova AN SSSR, 62 (1961).

* For continuous functions ($p = \infty$), these results are obtained with the aid of a relation between the operators U_n^* and U_n, established by V. F. Vlasov.

Note: Figure translations are in progress. See original paper for figures.

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