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**Abstract**

**Full Text**

*MATHEMATICS*

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## ON THE OSCILLATION OF SOLUTIONS OF SOME ORDINARY DIFFERENTIAL EQUATIONS

*(Presented by Academician I. G. Petrovskii, 20 XII 1961)*

In the present paper we consider the differential equation

$$\frac{d^m u}{dt^m} + F(u^2, t)u = 0, \quad (1)$$

where  $m \geq 2$ , and the function  $F(y, t)$  is continuous for  $0 \leq t, y < \infty$  and satisfies the following conditions:

$$F(y, t) \geq 0; \quad F(y_1, t) \leq F(y_2, t), \quad \text{when } y_1 < y_2. \quad (2)$$

We shall say that equation (1) is **strictly nonlinear** if, for large  $y$ ,  $F(y, t)$ , in addition to (2), satisfies the condition

$$\frac{F(y^2, t)}{\varphi(y)} \geq F(c^2, t), \quad c < 0, \quad (3)$$

where the function  $\varphi(y)$  is absolutely continuous in the interval  $(0, \infty)$  and

$$\varphi(y) > 0, \quad \varphi'(y) \geq 0, \quad \int_0^\infty \frac{dy}{y\varphi(y)} < \infty. \quad (4)$$

A solution  $u(t)$  of equation (1) is called **oscillatory** if it has an infinite number of zeros, and otherwise **nonoscillatory**. In the present paper some criteria for the oscillation of solutions of equation (1) are established.

**Lemma 1.** *Let  $u(t)$  be a continuous, nonnegative function in the interval  $(0, \infty)$ , and suppose it has continuous derivatives up to order  $m$  inclusive, which preserve their sign in this interval. If  $u^{(m)}(t) \leq 0$ , then there exists a number  $p$ ,  $0 \leq p \leq [\frac{m-1}{2}]$ , such that*

$$u^{(k)}(t) \geq 0 \quad (k = 0, 1, \dots, l),$$

$$(-1)^{m+k-1}u^{(k)}(t) \geq 0 \quad (k = l+1, \dots, m); \quad (5)$$

$$0 \leq u^{(l)}(t) \leq \frac{l!}{t^l}u(t), \quad (6)$$

where

$$l = 2p + \frac{1 + (-1)^m}{2}.$$

Let us indicate how the second part of the lemma is proved. According to (6), the function  $u^{(l)}(t)$  is nonnegative and nonincreasing. Therefore we have the inequality  $tu^{(l)}(t) \leq u^{(l-1)}(t)$ , by virtue of which, from the identities

$$tu^{(l-k)}(t) = (1+k)u^{(l-k-1)}(t) - (1+k)u^{(l-k-1)}(0) - \int_0^t [ku^{(l-k)}(\tau) - \tau u^{(l-k+1)}(\tau)] d\tau \quad (k = 1, 2, \dots, l-1)$$

we prove by induction that

$$tu^{(l-k)}(t) \leq (1+k)u^{(l-k-1)}(t) \quad (k = 1, 2, \dots, l-1),$$

whence inequality (6) follows immediately.

Let  $u(t)$  be some solution of equation (1). If  $u(t)$  or  $-u(t)$  satisfies inequalities (5), we shall say that  $u(t)$  belongs to the class  $A_p$ .

From the lemma formulated above it is clear that every nonoscillatory solution of equation (1) belongs to one of the classes  $A_p$ ,

$$0 \leq p \leq \left[ \frac{m-1}{2} \right].$$

**Theorem 1.** *If  $m$  is odd, then, in order that all solutions of equation (1) belonging to the class  $A_0$  tend to zero as  $t \rightarrow \infty$ , it is necessary and sufficient that*

$$\int_0^\infty F(c^2, t)t^{m-1} dt = \infty \quad (7)$$

for every  $c > 0$ .

**Proof.** If  $u(t) \in A_0$ ,  $u(t) > c > 0$  for  $t \geq 0$ , then, taking (5) into account, from the equality

$$\sum_{k=0}^{m-1} (-1)^k A_{m-1}^k t^{m-k-1} u^{(m-k-1)} + \int_{t_0}^t F(u^2, \tau) \tau^{m-1} u d\tau = c_0$$

we find

$$\int_{t_0}^{\infty} F(c^2, t)t^{m-1} dt < \frac{c_0}{c},$$

which contradicts condition (7). The contradiction obtained shows that if the integral

$$\int_{t_0}^{\infty} F(c^2, t)t^{m-1} dt$$

diverges for every  $c > 0$ , then all solutions of equation (1) belonging to the class  $A_0$  tend to zero as  $t \rightarrow \infty$ .

It can be shown that if, for some  $c > 0$ , the integral

$$\int_{t_0}^{\infty} F(c^2, t)t^{m-1} dt$$

converges, then the integral equation

$$u(t) = \frac{c}{2} + \frac{1}{(m-1)!} \int_t^{\infty} (\tau - t)^{m-1} F(u^2, \tau) u(\tau) d\tau$$

and, consequently, the differential equation (1) has a solution  $u(t)$  tending to a finite, nonzero limit as  $t \rightarrow \infty$ . Taking into account that  $u(t) \in A_0$ , the necessity of condition (7) becomes obvious.

For  $m = 3$  and  $F(y, t) \equiv F(t)$ , Theorem 1 implies Villari's theorem (1). The following theorem is proved in an analogous way.

**Theorem 2.** *If  $m$  is even, then, in order that all solutions of equation (1) belonging to the class  $A_0$  be bounded, it is necessary and sufficient that condition (7) hold for every  $c > 0$ .*

For the case  $m = 2$  the analogous theorem belongs to Nehari (2).

Following Kneser (3), we shall say that **condition (A)** is satisfied for equation (1) if equation (1) has an oscillatory solution and every nonoscillatory solution tends monotonically to zero as  $t \rightarrow \infty$ .

It is clear from the lemma that, in the case of even  $m$ , the fulfillment of condition (A) means that all solutions of equation (1) are oscillatory.

**Theorem 3.** *Let  $\varphi(t)$  be some function satisfying conditions (4). If, for every  $c > 0$ ,*

$$\int_{t_0}^{\infty} F(c^2, t) \frac{t^{m-1}}{\varphi(t)} dt = \infty, \quad (8)$$

*then condition (A) is satisfied for equation (1).*

**Proof.** Let  $m$  be odd,  $u(t) \in A_0$ . According to Theorem 1,  $\lim_{t \rightarrow \infty} u(t) = 0$ . Thus, to prove the theorem it is enough to show that for even  $m$  every solution is oscillatory, while for odd  $m$  it either belongs to the class  $A_0$ , or is oscillatory.

Suppose the contrary. Let  $u(t) \in A_p$ ,  $p \geq \frac{1 + (-1)^{m-1}}{2}$ . Without loss of generality, we may assume that  $u(t) > 0$  and all its derivatives up to order  $m - 1$  inclusive preserve their sign on the interval  $(0, \infty)$ .

Multiplying equation (1) by  $t^{m-1}/\varphi(t)u$  and integrating, we find

$$\frac{f(t)}{\varphi(t)u} - \int_{t_0}^t f(\tau) d(\varphi(\tau)u)^{-1} + \int_{t_0}^t F(u^2, \tau) \frac{\tau^{m-1}}{\varphi(\tau)} d\tau = c_1 + A_{m-1}^{m-l} \int_{t_0}^t \frac{\tau^{l-1}u^{(l)}}{\varphi(\tau)u} d\tau, \quad (9)$$

where

$$f(t) = \sum_{k=0}^{m-l-1} (-1)^k A_{m-1}^k t^{m-k-1} u^{(m-k-1)}.$$

By virtue of (5) and (6), from (9) it follows that

$$\int_{t_0}^t F(u^2, \tau) \frac{\tau^{m-1}}{\varphi(\tau)} d\tau \leq c_1 + (m-1)! l \int_{t_0}^{\infty} \frac{d\tau}{\tau\varphi(\tau)} = c_2. \quad (10)$$

Since  $u(t) > c > 0$ , hence we obtain

$$\int_{t_0}^{\infty} F(c^2, \tau) \frac{\tau^{m-1}}{\varphi(\tau)} d\tau > \infty,$$

which contradicts condition (8).

The result obtained from Theorem 3 for the equation

$$\frac{d^m u}{dt^m} + F(t)u = 0, \quad (1_0)$$

is a generalization of the theorems of Kneser <sup>(3)</sup>, Villari <sup>(1)</sup>, and Anan' eva-Balaganskii <sup>(4)</sup>.

Applying the results obtained by V. A. Kondrat' ev in <sup>(5)</sup> for equation (1<sub>0</sub>), one can prove the following theorem:

**Theorem 4.** 1) If

$$\int_{t_0}^{\infty} F(t) \frac{t^{m-1}}{\varphi(t)} dt = \infty, \quad F(t) \geq 0,$$

then condition (A) is satisfied for equation (1<sub>0</sub>).

2) If

$$\int_{t_0}^{\infty} |F(t)|t^{m-1} dt > \infty,$$

then the solutions of equation (1<sub>0</sub>) are nonoscillatory.

3) If

$$\int_{t_0}^{\infty} F(t) \frac{t^{m-1}}{\varphi(t)} dt = -\infty, \quad m \geq 3, \quad F(t) \leq 0,$$

then there exists a fundamental system consisting of

$$\frac{3 + (-1)^m}{2}$$

nonoscillatory and

$$m - \frac{3 + (-1)^m}{2}$$

oscillatory solutions.

For  $\varphi(t) \equiv t$  this theorem is proved in V. A. Kondrat'ev's work (5).

**Theorem 5.** If equation (1) is strictly nonlinear, then condition (7) is necessary and sufficient for condition (A) to hold for equation (1).

**Proof.** Necessity follows from Theorems 1 and 2. Let us prove sufficiency. Suppose the contrary. Let, under condition (7), equation (1) have a solution  $u(t) \in A_p$ ,  $p \geq \frac{1 + (-1)^{m-1}}{2}$ . It is clear that  $\lim_{t \rightarrow \infty} u(t) = \infty$ ; therefore, for large  $t$ , (3) will be satisfied.

From equation (1) we easily find

$$\frac{f(t)}{u\varphi(u)} - \int_{t_0}^t f(\tau) d(u\varphi(u))^{-1} + \int_{t_0}^t \frac{F(u^2, \tau)}{\varphi(u)} \tau^{m-1} d\tau = c_3 + A_{m-1}^{m-l} \int_{t_0}^t \frac{\tau^{l-1} u^{(l)}}{u\varphi(u)} d\tau. \quad (11)$$

Taking into account (3), (5), and (6), from (11) we obtain

$$\int_{t_0}^{\infty} F(c^2, \tau) \tau^{m-1} d\tau \leq c_3 + (m-1)! \int_{u(t_0)}^{\infty} \frac{du}{u\varphi(u)} < \infty,$$

which contradicts condition (7).

When  $m = 2$ , Theorem 5 yields the theorems of Atkinson <sup>6</sup> and Jones <sup>7</sup>.

We shall say that for equation (1) **condition (B)** is satisfied if every solution  $u(t)$  either oscillates, or

$$\lim_{t \rightarrow \infty} u^{(m-1)}(t) = 0.$$

The following theorem is easily proved:

**Theorem 6.** *For condition (B) to hold, it is necessary and sufficient that*

$$\int^{\infty} F(c^2 t^{2m-2}, t) t^{m-1} dt = \infty$$

*for every  $c > 0$ .*

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