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Abstract

Full Text

MATHEMATICS

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ON THE APPLICATION OF A REGULARITY ESTABLISHED BY WIMAN AND VALIRON TO THE STUDY OF CHARACTERISTIC FUNCTIONS OF PROBABILITY LAWS

(Presented by Academician S. N. Bernstein on 21 XI 1961)

I. Marcinkiewicz proved ⁽⁵⁾ that an entire function of finite order ρ with exponent of convergence of zeros $\rho_1 < \rho$ can be a characteristic function of a probability law (c.f.) only in the case when $\rho \leq 2$. In particular, the following theorem is valid:

An entire function of the form $\exp f(z)$, where $f(z)$ is a polynomial, is a c.f. only when $f(z) = -\gamma z^2 + i\beta z$ ($\gamma \geq 0$, β real).

Known examples of c.f.'s of the form $\exp f(z)$, where $f(z)$ is an entire function not below normal type of order 1, include, as the simplest such example, the c.f. of the Poisson law $\exp\{\lambda(e^{i\mu z} - 1)\}$, where $\lambda > 0$, and $\mu \neq 0$ is real. Yu. V. Linnik posed the question ^(?), p. 255): can $\exp f(z)$ be a c.f. if $f(z)$ is an entire transcendental function of not more than minimal type of order 1.

From the definition of a c.f. it is readily obtained that for an entire c.f. $\varphi(z)$ the relation $|\varphi(x + iy)| \leq |\varphi(iy)|$, $-\infty < x, y < \infty$, must hold, which Yu. V. Linnik called the "ridge property." Thus, if $\exp f(z)$ is a c.f., then it is necessary that $\operatorname{Re} f(x + iy) \leq \operatorname{Re} f(iy)$, $-\infty < x, y < \infty$.

Theorem 1. *If an entire transcendental function $f(z)$ satisfies the relation*

$$\operatorname{Re} f(x + iy) \leq M(|y|), \quad -\infty < x, y < \infty, \quad (1)$$

where $M(r) = \max_{|z|=r} |f(z)|$, then

$$\overline{\lim}_{r \rightarrow \infty} r^{-1} \ln M(r) > 0.$$

Thus the answer to Yu. V. Linnik's question is negative. The proof is based on a regularity discovered by Wiman and Valiron: every entire function $f(z)$ in a sufficiently small neighborhood of a point ζ , where $|f(\zeta)| = M(|\zeta|)$, admits an asymptotic representation of the form $f(z) \sim (z/\zeta)^N f(\zeta)$. We were unable to use the results of Wiman and Valiron themselves. We used several other results

of the same nature, obtained by means of a certain improvement of Macintyre's method (4).

1°. We rely essentially on the results of B. Ya. Levin on the refined order of growth of an entire function. We shall state these results.

Let $f(z)$ be an entire function of order ρ . One can find (see (1), pp. 52-60) a function $\rho(r)$, called by B. Ya. Levin a strong refined order of the function $f(z)$, such that: a) $\ln M(r) \leq r^{\rho(r)}$ for all r , $0 \leq r < \infty$; b) $\ln M(r) = r^{\rho(r)}$ for some unbounded set of values of r ; c)

$$\rho(r) = \rho + \{\vartheta_1(\ln r) - \vartheta_2(\ln r)\}(\ln r)^{-1},$$

where $\vartheta_1(t)$ and $\vartheta_2(t)$, $-\infty < t < \infty$, are continuously differentiable convex functions with piecewise continuous second derivative. Each of the functions $\vartheta_\alpha(t)$, $\alpha = 1, 2$, is either identically 0, or satisfies the conditions: 1) $\lim_{t \rightarrow \infty} \vartheta_\alpha(t) = \infty$, 2) $\lim_{t \rightarrow \infty} \vartheta'_\alpha(t) = 0$, 3) $\lim_{t \rightarrow \infty} \vartheta''_\alpha(t) \{\vartheta'_\alpha(t)\}^{-1} = 0$. From the construction of $\rho(r)$ it readily follows that if $f(z)$ is of maximal type, then one may assume $\vartheta_1 \neq 0$, $\vartheta_2 \equiv 0$. If $f(z)$ —

of minimal type, then one may assume $\vartheta_1 \equiv 0$, $\vartheta_2 \neq 0$ *. It is easy to show that in the latter case one may impose on ϑ_2 the additional condition

$$\vartheta_2(t) \geq (1 + t^2)^{-1}.$$

Put $\Psi(t) = \ln M(e^t)$, $\Phi(t) = \exp\{\rho t + \vartheta_1(t) - \vartheta_2(t)\}$. We shall have: a) $\Psi(t) \leq \Phi(t)$ for all t , $-\infty < t < \infty$; b) $\Psi(t) = \Phi(t)$ for some set E of values of t unbounded on the right. Since $\Psi(t)$ is convex, for $t \in E$ it is differentiable and $\Psi'(t) = \Phi'(t)$. We note that if $f(z)$ is transcendental, then

$$\lim_{t \rightarrow \infty} \Psi'(t) = \infty.$$

2°. Denote by ζ that point of the circle $|z| = e^t$, $t \in E$, where $|f(z)| = M(e^t)$. We shall study the behavior of $f(z)$ in a neighborhood of ζ . Put

$$N_k = N_k(t) = (k!)^{-1} (d/d \ln \zeta)^k \ln f(\zeta).$$

Lemma 1. For $t \in E$

$$N_1(t) = \Psi'(t) = \Phi'(t), \tag{2}$$

$$\operatorname{Re} N_2(t) \leq \frac{1}{2} \max[\Phi''(t-0), \Phi''(t+0)]. \tag{3}$$

Let R be some positive number; denote

$$\Delta = \Delta(t, R) = \Phi(t + \varepsilon R) - \Phi(t) - \Phi'(t)\varepsilon R,$$

where $\varepsilon = \varepsilon(t, R) = \pm 1$ is chosen so that the expression on the right is the largest; evidently, for $t \in E$

$$\Delta \geq \Psi(t + \varepsilon R) - \Psi(t) - \Psi'(t)\varepsilon R \geq 0. \tag{4}$$

Theorem 2. For $0 < \theta < 1$, $|\tau| \leq \theta R$, the relation

$$f(\zeta e^\tau) = f(\zeta) e^{N_1 \tau} \{1 + \omega(\tau)\}, \quad (5)$$

holds, where

$$|\omega(\tau)| \leq \theta^2 (e^{2\Delta} - 1) (e^\Delta - \theta^2)^{-1}. \quad (6)$$

Proof uses (2), (4) and almost entirely coincides with the proof of the main theorem of Macintyre's paper (4).

Corollary 1. If $\theta < e^{-\Delta/2}$, then for $|\tau| \leq \theta R$ the function $f(\zeta e^\tau)$ does not vanish.

Corollary 2. For $k = 2, 3, 4, \dots$ the estimate

$$|N_k(t)| \leq 3kR^{-k} e^{k\Delta/2}$$

holds.

Theorem 3. For $0 < \theta \leq \frac{1}{2} e^{-\Delta/2}$, $|\tau| \leq \theta R$, $n = 2, 3, 4, \dots$, the relation

$$f(\zeta e^\tau) = f(\zeta) \exp[N_1 \tau + N_2 \tau^2 + \dots + N_n \tau^n] \{1 + \omega_n(\tau)\}, \quad (5')$$

holds, where **

$$|\omega_n(\tau)| \leq An(e^\Delta \theta)^{n+1}.$$

Proof. Put

$$\chi(\tau) = f(\zeta e^\tau) |f(\zeta)|^{-1} \exp[-N_1 \tau - \dots - N_n \tau^n].$$

By Corollary 1, for $|\tau| \leq \theta R$

$$\ln \chi(\tau) = N_{n+1} \tau^{n+1} + N_{n+2} \tau^{n+2} + \dots,$$

whence

$$|\ln \chi(\tau)| \leq \sum_{k=n+1}^{\infty} |N_k(t)| (\theta R)^k.$$

* The proof of this assertion (which is used by us only in the consideration of functions of exactly minimal type of order 1) was communicated to me by B. Ya. Levin.

** We agree to denote by the letter A (various) absolute constants.

We estimate the sum of the series on the right with the aid of Corollary 2. We shall have

$$|\ln \chi(\tau)| \leq 12(n+1)(\theta e^{\Delta/2})^{n+1} \leq 5,$$

whence

$$|\omega_n(\tau)| = |\exp(\ln \chi(\tau)) - 1| \leq 12(n+1)(\theta e^{\Delta/2})^{n+1} e^5.$$

3°. In this subsection we shall formulate the lemmas needed for the proof of Theorem 1. The validity of these lemmas is verified by means of simple, but rather cumbersome, calculations. In what follows we shall consider only values of t from E ; in accordance with this we agree to understand the symbol $\lim_{t \rightarrow \infty}$. We shall also agree henceforth to put $R = R(t) = \{\Phi'(t)\}^{-1/2}$ and to regard the function $f(z)$ as transcendental. By the latter assumption,

$$\lim_{t \rightarrow \infty} R(t) = 0.$$

Lemma 2*. The relation

$$\lim_{t \rightarrow \infty} \Delta(t, R(t)) = \frac{1}{2}\rho$$

is valid.

Lemma 3. Let the function $f(z)$ satisfy the condition

$$\lim_{r \rightarrow \infty} r^{-1} \ln M(r) = 0.$$

If the order of $f(z)$ is equal to 1, we shall assume that $\vartheta_1 \equiv 0$, $\vartheta_2 \geq (1+t^2)^{-1}$.

Let $k = k(t)$, $\lambda = \lambda(t)$ be real functions, $0 < k < 1$, $kR < |\lambda| < R$. Then, for all sufficiently large t ($\in E$),

$$\begin{aligned} \Phi(t + \ln \cos \lambda) - \Phi(t) + \frac{1}{2} \max[\Phi''(t-0), \Phi''(t+0)] < \\ < \frac{1}{3} k^2 (\rho - 1 - \vartheta_2'(t)). \end{aligned}$$

Lemma 4. Let $0 < \theta < 1$, and let a and b be arbitrary real numbers, with $a \neq 0$. Denote by Ω the interval

$$\frac{1}{8}\theta|a|^{-1/2} \leq x \leq \theta|a|^{-1/2}.$$

Then

$$\max_{\Omega} |ax + bx^2| - \min_{\Omega} |ax + bx^2| \geq \frac{1}{16}\theta|a|^{1/2}.$$

4°. **Proof of Theorem 1.** Put

$$\theta = \theta(t) = 200\{\Phi'(t)\}^{-1/2};$$

by Lemma 2, for all sufficiently large t the condition $\theta \leq \frac{1}{2}e^{-\Delta/2}$ will be satisfied. Choose $\tau = \tau(t)$ equal to $i\lambda(t)$, where $\lambda = \lambda(t)$ is real and satisfies the conditions: 1) λ is positive if $\pi/2 \leq \arg \zeta < \pi$ or $3\pi/2 \leq \arg \zeta < 2\pi$, and negative if $0 \leq \arg \zeta < \pi/2$ or $\pi \leq \arg \zeta < 3\pi/2$; 2) $\frac{1}{8}\theta R \leq |\lambda| \leq \theta R$; 3) $N_1\lambda - (\text{Im } N_2)\lambda^2 \equiv -\arg f(\zeta) \pmod{2\pi}$. The possibility of such a choice of λ follows easily from Lemma 4. By Theorem 2 ($n = 2$) we obtain

$$f(\zeta e^\tau) = |f(\zeta)| \exp[-\text{Re } N_2 \cdot \lambda^2] \{1 + \omega_2(\tau)\},$$

whence

$$\text{Re } f(\zeta e^\tau) \geq |f(\zeta)| \exp[-\text{Re } N_2 \cdot \lambda^2] \{1 - A\theta^3 e^{3\Delta/2}\}.$$

In the left-hand side of this inequality, in view of (1), one may put $M(e^t \cos \lambda)$. In the right-hand side one may replace $\text{Re } N_2$ by the expression from (3). Taking logarithms of the resulting inequality, we arrive at the relation

$$\begin{aligned} \Phi(t + \ln \cos \lambda) - \Phi(t) + \frac{1}{2} \max\{\Phi''(t-0), \Phi''(t+0)\} &\geq \\ &\geq \ln\{1 - A\theta^3 e^{3\Delta/2}\}. \end{aligned} \tag{7}$$

* From this lemma there follow estimates of the remainder term ω in Theorems 2 and 3 that do not depend on t ; therefore its combination with Theorems 2 and 3 is of independent interest. If, following Macintyre, one puts $R = \{\Psi(t)\}^{-1/2}$ ($= \{\Phi(t)\}^{-1/2}$), then one obtains $\lim_{t \rightarrow \infty} \Delta = \frac{1}{2}\rho^2$, whence there follows an estimate of ω more precise than in (4). The most important difference for what follows between our results and the results (4) is the knowledge that the points ζ in (5) and (5') lie on every circle $|z| = e^t$, $t \in E$. The advantage of the results (4) is that they are applicable to functions of finite and lower order.

Suppose now that $\lim_{r \rightarrow \infty} r^{-1} \ln M(r) = 0$. Then, applying Lemmas 2 and 3, from (7) we conclude that for all sufficiently large t ($\in E$) the relation

$$1 - \rho + (1 + t^2)^{-1} \leq A\{\Phi'(t)\}^{-1/2}.$$

holds.

But the latter cannot hold. Indeed, if $\rho < 1$, then the limit of the left-hand side as $t \rightarrow \infty$ is positive, whereas the right-hand side is equal to zero. If, however,

$\rho = 1$, then the right-hand side ($< e^{-t/4}$) tends to zero much faster than the left-hand side.

Remark. It is not difficult to show that the assertion of Theorem 1 remains valid if condition (1) is satisfied only for $|x| \geq A$, $|y| \geq A$, where A is some constant.

5°. The following generalizations of Theorem 1 are, it seems to us, of independent interest.

Theorem 4. Let $F(w)$ be an entire function such that, for all sufficiently large values of $|w|$, $|F(w)| \leq F(|w|)$. Let $g(z)$ be an entire function for which it is known that the superposition $f(z) = F(g(z))$ satisfies condition (1). Then either $g(z)$ is a polynomial of second degree, or

$$\lim_{r \rightarrow \infty} r^{-1} \ln M(r, g) > 0.$$

This theorem also generalizes a result of E. Lukacs ⁽³⁾.

Theorem 5. Let $f(z)$ be an entire (transcendental) function of order ρ , and let $g(z)$ be an entire function satisfying the condition

$$\overline{\lim}_{r \rightarrow \infty} (\ln r)^{-1} \ln \ln \ln M(r, g) < \rho, \quad \text{if } \rho > 0,$$

$$\overline{\lim}_{r \rightarrow \infty} (\ln r)^{-1} \ln \ln M(r, g) < \infty, \quad \text{if } \rho = 0.$$

If the relation $(-\infty < x, y < \infty)$

$$\operatorname{Re} f(x + iy) + \ln |g(x + iy)| \leq M(|y|, f) + \ln M(|y|, g), \quad (8)$$

holds, then

$$\overline{\lim}_{r \rightarrow \infty} r^{-1} \ln M(r, f) > 0.$$

Relation (8) is always satisfied if $g(z) \exp f(z)$ is a ch.f. Thus, Theorem 5 strengthens a result of Marcinkiewicz ⁽⁵⁾.

The proofs of Theorems 4 and 5 differ from the proof of Theorem 1 only in certain specific features of the choice of λ .

I express my deep gratitude to B. Ya. Levin for his attention to the work.

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