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Abstract

Full Text

MATHEMATICS

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ON A GENERALIZED CAUCHY FORMULA FOR AXISYMMETRIC VECTOR FIELDS

(Presented by Academician I. N. Vekua on 12 IV 1962)

We consider the complex differential equation

$$\frac{\partial f}{\partial \bar{z}} - \frac{1}{4ir} f - \frac{1}{4ir} \bar{f} = 0, \quad f = V_r + iV_x, \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial r} \right), \quad (1)$$

equivalent to the system of two real equations

$$\frac{\partial}{\partial x}(rV_x) + \frac{\partial}{\partial r}(rV_r) = 0, \quad \frac{\partial V_r}{\partial x} - \frac{\partial V_x}{\partial r} = 0;$$

the latter express the potentiality of a spatial axisymmetric vector field, given in a cylindrical coordinate system by the components (V_x, V_r) , and the incompressibility of the medium. On the basis of the results of the paper ⁽³⁾ (see also ⁽⁴⁾) and using the methods of the work of I. N. Vekua ⁽¹⁾ (§ 10), kernels of a generalized Cauchy formula are constructed for solutions of equation (1) in the meridian half-plane $r \geq 0$.

1. The main result is contained in the theorem:

Theorem 1. *There exist and are uniquely determined two functions $U(z_0, \bar{z}_0; z, \bar{z})$, $V(z_0, \bar{z}_0; z, \bar{z})$ (kernels of the generalized Cauchy formula), depending on the coordinates of two points $P(x, r)$, $P_0(x_0, r_0)$, $z = x + ir$, $z_0 = x_0 + ir_0$, and satisfying the conditions:*

I. For fixed z_0 , $r_0 > 0$, the functions U, V with respect to z, \bar{z} are defined and single-valued in the entire upper half-plane $r \geq 0$ with the point $z = z_0$ removed; in a neighborhood of the point $z = z_0$ the function U has principal part

$$-\frac{1}{z - z_0} \frac{\sqrt{\bar{z}_0 - z_0}}{\sqrt{-z_0}} \frac{\sqrt{-z}}{\sqrt{\bar{z} - z}}, \quad (2)$$

the remaining terms of the expansion, as well as the function V , have at the point $z = z_0$ singularities of no higher than logarithmic order.

II. For fixed z , $r > 0$, the functions U, V with respect to z_0, \bar{z}_0 are defined and single-valued in the entire upper half-plane $r_0 \geq 0$ with the point $z_0 = z$ removed; in a neighborhood of the point $z_0 = z$ they have the behavior indicated in I.

III. For fixed z , the functions U, V depend analytically on the variables z_0, \bar{z}_0 for $z_0 \neq z$ and satisfy the system of equations

$$\frac{\partial U}{\partial \bar{z}_0} - AU - \bar{B}V = 0, \quad \frac{\partial V}{\partial \bar{z}_0} - AV - \bar{B}U = 0, \quad (3)$$

$$A(z_0, \bar{z}_0) \equiv B(z_0, \bar{z}_0) \equiv -\frac{1}{2}(z_0 - \bar{z}_0)^{-1}, \quad z_0 \neq z.$$

IV. For a fixed value z_0 , the functions U, V depend analytically on the variables z, \bar{z} for $z \neq z_0$ and satisfy the system of equations

$$\frac{\partial U}{\partial \bar{z}} + AU + BV = 0, \quad \frac{\partial \bar{V}}{\partial \bar{z}} + AV + B\bar{U} = 0,$$

$$A(z, \bar{z}) \equiv B(z, \bar{z}) \equiv -\frac{1}{2}(z - \bar{z})^{-1}, \quad z \neq z_0. \quad (4)$$

V. For fixed values of z_0 , the functions U, V preserve regularity and analyticity in z, \bar{z} up to the axis $r = 0$, and their limiting values satisfy the identity

$$U(z_0, \bar{z}_0; x, x) + V(z_0, \bar{z}_0; x, x) \equiv 0, \quad -\infty \leq x \leq \infty; \quad (5)$$

as $|z| \rightarrow \infty$, the functions U, V tend to zero.

VI. For fixed values of z , the functions U, V preserve regularity and analyticity in z_0, \bar{z}_0 up to the axis $r_0 = 0$, and their limiting values satisfy the identities

$$U(x_0, x_0; z, \bar{z}) \equiv V(x_0, x_0; z, \bar{z}) \equiv 0, \quad -\infty \leq x_0 \leq \infty; \quad (6)$$

as $|z_0| \rightarrow \infty$, the functions U, V tend to zero.

2. Let Γ denote the aggregate of a finite number of rectifiable contours lying in the half-plane $r \geq 0$. Suppose that these contours have no common points and no points of self-intersection. Some of them may lie above the axis $r = 0$, and then they represent closed or open* simple curves. The others may have common points with the axis $r = 0$, and then they represent simple arcs resting with their endpoints on the axis $r = 0$. Let Γ , together with certain segments Γ_0 of the x -axis, bound a connected set G in the upper half-plane $r \geq 0$. The domain G may be either finite or infinite.

Theorem 2. Let $f(x, r)$ be a solution of equation (1), regular inside G and, say, bounded up to $\Gamma + \Gamma_0$. Then:

1) If G is finite, then at each interior point $z \in G$ we have

$$f(x, r) = \frac{1}{2\pi i} \int_{\Gamma} U(z_0, \bar{z}_0; z, \bar{z}) f(x_0, r_0) dz_0 - \overline{V(z_0, \bar{z}_0; z, \bar{z}) f(x_0, r_0)} d\bar{z}_0. \quad (7)$$

2) If G is infinite and $f(x, r)$ has at infinity the uniformly attainable limit iV_x^∞ , then for $z \in G$

$$f(x, r) = iV_x^\infty + \frac{1}{2\pi i} \int_{\Gamma} U(z_0, \bar{z}_0; z, \bar{z}) f(x_0, r_0) dz_0 - \overline{V(z_0, \bar{z}_0; z, \bar{z}) f(x_0, r_0)} d\bar{z}_0. \quad (8)$$

3) For any density $\mu(z_0, \bar{z}_0)$ summable on Γ , the generalized Cauchy-type integral

$$w(z, \bar{z}) = \frac{1}{2\pi i} \int_{\Gamma} U(z_0, \bar{z}_0; z, \bar{z}) \mu(z_0, \bar{z}_0) dz_0 - \overline{V(z_0, \bar{z}_0; z, \bar{z}) \mu(z_0, \bar{z}_0)} d\bar{z}_0 \quad (9)$$

represents a solution of equation (1), piecewise regular in the entire closed upper half-plane $r \geq 0$; the set of discontinuity points of w lies on Γ ; at each interior point $x \in \Gamma_0$ the function $w(x, x)$ is analytic in x and $\operatorname{Re} w(x, x) = 0$; as $|z| \rightarrow \infty$, the function (9) tends to zero.

The first two assertions follow from identities (1), (3) and formula (2), if one uses Green's formulas (in complex form), applied—

* In the latter case the contour is regarded as consisting of two banks.

with respect to the equation (1) under consideration, with an additional limiting transition to the axis $r = 0$. The third assertion follows from the identities (4) and (5).

3. Let us also consider the equation conjugate to equation (1):

$$\frac{\partial f'}{\partial z} + \frac{1}{4ir} f' - \frac{1}{4ir} \bar{f}' = 0. \quad (10)$$

Theorem 3. Let $f'(x, r)$ be a solution of equation (10), regular inside G , say bounded up to Γ , and satisfying the condition $f'(x, x) \equiv 0$ for $x \in \Gamma_0$. Then:

1) If G is finite, then for $z \in G$

$$f'(x, r) = -\frac{1}{2\pi i} \int_{\Gamma} U(z, \bar{z}; z_0, \bar{z}_0) f'(x_0, r_0) dz_0 - V(z, \bar{z}; z_0, \bar{z}_0) \overline{f'(x_0, r_0)} d\bar{z}_0. \quad (11)$$

- 2) If G is infinite and the ratio f'/r has a uniformly attainable limit iV_x^∞ at infinity, then for $z \in G$

$$f'(x, r) = iV_x^\infty r - \frac{1}{2\pi i} \int_{\Gamma} U(z, \bar{z}; z_0, \bar{z}_0) f'(x_0, r_0) dz_0 - V(z, \bar{z}; z_0, \bar{z}_0) \overline{f'(x_0, r_0)} d\bar{z}_0. \quad (12)$$

- 3) The generalized Cauchy-type integral for equation (10) has the form

$$w'(z, \bar{z}) = -\frac{1}{2\pi i} \int_{\Gamma} U(z, \bar{z}; z_0, \bar{z}_0) \mu(z_0, \bar{z}_0) dz_0 - V(z, \bar{z}; z_0, \bar{z}_0) \overline{\mu(z_0, \bar{z}_0)} d\bar{z}_0 \quad (13)$$

for any density μ summable on Γ , and represents a solution of equation (10), piecewise regular in the entire closed upper half-plane; the set of discontinuity points of w' lies on Γ ; at each interior point $x \in \Gamma_0$, the function (13) satisfies the condition $w'(x, x) \equiv 0$; as $|z| \rightarrow \infty$ it tends to zero.

Theorem 3 is a consequence of the fact that the functions $U'(z_0, \bar{z}_0; z, \bar{z}) = -U(z, \bar{z}; z_0, \bar{z}_0)$, $V'(z_0, \bar{z}_0; z, \bar{z}) = -V(z, \bar{z}; z_0, \bar{z}_0)$ constitute the kernels of the generalized Cauchy formula for the conjugate equation (10). Let us also note that the last assertion follows easily from the identities (3), (6).

4. The generalized Cauchy formula for differential equations of the form (1) was constructed in [1] under the assumption that the coefficients of the equation are continuous; in the important special case when the coefficients are analytic in x, r and regular in G , another method for constructing these kernels was indicated in § 10 of the same article, based on continuation of solutions into the domain of complex values of the arguments. Later (see [2]) the theory was extended also to the case when the coefficients of the equation belong to $L_p(G)$, $p > 2$.

Equation (1) satisfies none of the indicated assumptions if the domain G has the axis $r = 0$ (or a part of it) as its boundary. However, thanks to the simple structure of the coefficients of equation (1), the method of analytic continuation from [1], § 10, is applicable to it (and also to more general equations).

In the proof of Theorem 1 the results of article [3] and the parallel theory of equation (10) are used essentially. In addition, one also considers the equation adjoint to (1), which differs from (1) by the sign before the second coefficient, and not before the first, as equation (10) does:

$$\frac{\partial \hat{f}}{\partial z} - \frac{1}{4ir} \hat{f} + \frac{1}{4ir} \bar{f} = 0. \quad (14)$$

The system (4), as is easy to see, is equivalent to the pair of equations (1), (14). The starting point for constructing the kernels U, V is the fundamental solutions

of equations (1), (14), vanishing at infinity. In constructing the elementary solutions themselves, each step is quite effective and reduces to differentiation and integration (in the complex domain). Along the way there also arises a certain singular integral equation, which can be effectively solved by the Wiener-Hopf method. The same degree of effectiveness is retained at the final stages of constructing the kernels U, V .

Every vector field f, \hat{f} satisfying equations (1), (14), respectively, admits its own potential $\varphi, \hat{\varphi}$; moreover, both of these functions, as is easy to verify, are certain axisymmetric potentials. For the elementary solutions mentioned above, $\varphi \neq \hat{\varphi}$ (one of these potentials is multivalued on the plane with the point $z = z_0$ removed), whence it follows that, for constructing the kernels U, V , knowledge of the classical fundamental solution in the theory of axisymmetric potentials is insufficient.

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